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RESEARCH REPORT No. MH-7

Statistical Mechanics of Stationary Homogeneous Hydromagnetic Turbulence

ROBERT H. KRAICHNAN

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Robert H. Kraichnan

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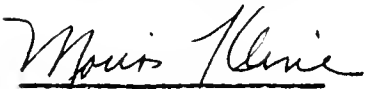
Morris Kline
Project Director

April, 1957

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Abstract

The statistical mechanics of stationary, homogeneous, hydromagnetic turbulence is treated by a method which does not require the assumption of small departure from a state of detailed balance. The equations of motion under cyclic boundary conditions are expressed in terms of the external driving forces and linearly independent dynamic variables formed from the spatial Fourier coefficients of velocity and magnetic fields. The nonlinear interaction is exhibited as the sum of individually conservative elementary interactions among triads of modes. A functional equation is constructed giving necessary and sufficient conditions for a distribution of time functions to satisfy the equations of motion; it involves only the second-, third-, and fourth-order moments of the distribution functional. By using the fact that each mode interacts with many others, a procedure is developed in which each elementary interaction is introduced as a perturbation on the total motion. The moments appearing in the functional equation are expressed by means of this procedure in terms of the diagonal elements of the covariance matrices of the dynamic variables and external forces and the distribution-averaged infinitesimal impulse response matrix of the system. Closed equations for these quantities are obtained so that the power spectra in frequency and wave number of velocity and magnetic fields, and the energy exchange among the modes, may be determined from the driving force power-spectrum. A variational method is given for defining and constructing the most nearly normal distribution functional with the previously derived second-, third-, and fourth-order moments, thereby determining the higher moments. The consistency with the present theory is confirmed for the cascade hypothesis fundamental to the Kolmogorov similarity theory of the inertial range. It is found that the rate of direct energy-transfer from modes of wave number below k_1 to modes of wave number above $k_2 \gg k_1$ is asymptotically proportional to $\varepsilon(k_1/k_2)^{4/3}$, where ε is the total rate of energy dissipation per unit mass and k_1, k_2 are both within the inertial range.

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Corrections

The normalization of the integral transform $U(k,\tau)$ introduced in (9.8) is correctly given by

$$U(k,\tau) = (2\pi)^{-3} \int \hat{U}(x,\tau)\, e^{-i\underline{k}\cdot\underline{x}}\, d^3x \ .$$

In (A.11), instead of $-j()$ read $+j()$.

Additional Comments

The corroboration of the Kolmogorov similarity theory given in Section 9 has a doubtful significance. It should be noted that the universal form for the τ-dependence of $U(k,\tau)$ given in (9.12) is physically a rather implausible extension of dimensional reasoning. The convective action of the large eddies may be expected to affect the time-dependence at high wave numbers, so that even if $U(k,0)$ actually is a universal function in the equilibrium range, unaffected by direct interaction with the large eddies, $U(k,\tau)$ probably is not. (This effect of convection has been discussed by Kolmogorov [2] and Heisenberg [10].) Although (9.12) yields a consistent universal expression for $S(k,0)$, as indicated in the text, it does not follow from this that the expression for $S(k,\tau)$, $(\tau \neq 0)$, is independent of the low-lying wave numbers. Thus, the consistency of (9.12) with the full equations of the present theory has not been demonstrated.

Further consideration of the convective effects of the large eddies leads to the conclusion that the idealized model of the present paper, in which the energy-containing range is maintained by the action of homogeneous driving forces $\underline{F}(\underline{k},t)$ that are statistically independent for different $\underline{k}$'s, may differ from laboratory turbulence more significantly than was indicated in Section 10. In the model the dimensions of the system are very large compared to the reciprocal of a typical wave number in the principal energy-containing region, and the phases at adjacent wave numbers in this region are essentially unrelated in a typical realization of the process. In laboratory turbulence the localization of the system to dimensions comparable to eddy sizes in the energy-containing region (and/or the generation of the large eddies by fixed mechanical structures) indicates that the phase in this region varies only smoothly with wave number in a typical realization. Thus the low-lying Fourier modes in laboratory turbulence exhibit a species of coherence which is lacking in the model. The presence or absence of this coherence conceivably may affect interactions at the higher wave numbers by influencing the extent to which the low-lying Fourier modes effectively convect the high wave number region as a whole, or disturb internal phase relations in that region.

June 1957

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1. Introduction

At the Reynolds numbers characteristic of turbulence there exist, for given boundary conditions, a rich manifold of distinct solutions of the hydrodynamic equations of incompressible motion[1]. The individual solutions have very complicated structure and are unstable, going over into each other under the action of perturbations. We often are interested in properties of the motion which can be expressed as averages over times long compared with characteristic periods of the individual solutions. In view of the presence in any physical situation of irreducible random perturbations of the flow it is reasonable to assume that such properties can be identified with statistical mechanical averages over an appropriate distribution of individual solutions.

The real and imaginary parts of the spatial Fourier coefficients of the velocity field (and magnetic field, if present) may be regarded as phase-space coordinates analogous to the canonical coordinates and momenta of Hamiltonian systems. It then may be shown that the system obeys a Liouville equation similar to that for Hamiltonian systems[1]. There are essential differences, however, between a turbulent fluid and systems, such as a dilute gas of hard spheres, for which the Gibbs statistical mechanics has been most successful.

In a dilute gas of hard spheres, the particles are noninteracting almost all of the time, and the interaction can be well described in terms of transition probabilities, due to two-particle collisions, between states of noninteraction. The collisions represent a small time-averaged perturbation in the Hamiltonian of the uncoupled particles. In the turbulent fluid, a typical Fourier mode undergoes simultaneous and continuous interaction with a large number of other modes. The motion is due entirely to the interaction. For the dilute gas we may be interested principally in the velocity distribution of the particles and interested in the collisions only in terms of their frequency and scattering angles. For the fluid, since the interaction is continuous, we necessarily are interested in the behavior of the Fourier modes <u>during the interaction</u>. The detailed structure of the interaction cannot be ignored.

We may note also that stationary turbulence represents a very strong departure from a state of detailed balance. It follows from Kolmogorov's similarity considerations[2]* that the temperature characterising the distribution of energy among the individual modes in the inertial range varies with wave number as $k^{-11/3}$. Thus it cannot be expected that methods of irreversible statistical mechanics

*See also G.K. Batchelor[4].

based on small departure from detailed balance could be successful in describing the transport of energy across the wave number spectrum.

A third difference between a turbulent fluid and a dilute gas is that the Fourier modes of the fluid all enter differently into the equations of motion. The distribution function must exhibit a much smaller degree of degeneracy than that for a gas of identical particles.

G. I. Taylor[3], followed by many other authors[4], has formulated the statistical problem of turbulence as that of determining moments of the velocity distribution. The moments of greatest interest are contained in the space-time covariance tensor of the velocity; this determines the spectral distribution of the kinetic energy in wave number and frequency. Equations of motion for various distribution moments result from multiplying the Navier-Stokes equation with suitable functions and averaging. As a consequence of nonlinearity, the equation for the velocity covariance contains the third moments also, and direct attempts at elimination of the latter lead to a series of incomplete systems of equations involving successively higher moments.

A part of this grave difficulty seems due to incomplete formulation. For it is not to be expected that the Navier-Stokes equation determines a unique form of statistical distribution, even under the conditions of isotropy and stationarity, unless one specifies, directly or indirectly, the full statistical structure of the forces which supply energy to the turbulence. To make this plausible, we may note that in a conservative Hamiltonian system the specification of mean energy, and the imposition of stationarity, leave the higher moments largely undetermined, since the simultaneous distribution may be any function of the constants of motion. If dissipation and driving forces are introduced the ambiguity still remains, unless the full statistical distribution of the driving forces is fixed.

Whatever supplementary conditions are imposed on the distribution must be consistent with the equations of motion and appropriate to the physical situation to be represented.

2. Description of the present approach

The present paper developed from the following argument, which, for brevity, we state for nonmagnetic turbulence. Let the velocity field $\underset{\sim}{u}$ be represented in the usual manner by Fourier modes. The interaction among three modes $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{u}(\underset{\sim}{k}')$, $\underset{\sim}{u}(\underset{\sim}{k}-\underset{\sim}{k}')$ can proceed along a direct path, represented by terms

involving $\underset{\sim}{u}(\underset{\sim}{k}')\underset{\sim}{u}(\underset{\sim}{k}-\underset{\sim}{k}')$, $\underset{\sim}{u}(\underset{\sim}{k})u(\underset{\sim}{k}'-\underset{\sim}{k})$, $\underset{\sim}{u}(\underset{\sim}{k})\underset{\sim}{u}(-\underset{\sim}{k}')$ in the respective equations of motion of these modes, and along indirect paths involving chains of interactions with the other excited modes. In view of the mixing action characteristic of turbulence let us suppose that the combined action of all the indirect interactions does not convey phase information among the three modes, so that the triple correlation among them is due to the presence of the direct interaction. A pertinent analogy would be the mutual phase relations induced by the interaction of three nonlinear oscillators. If many modes are excited and interacting, the direct interaction should represent a small disturbance of the motion of the three modes, and then it should be possible to calculate by perturbation methods the triple correlations which appear in the equation of motion of the spectrum tensor.

In the following sections it is shown that a self-consistent procedure of this sort can be carried out in which each elementary interaction among three Fourier modes is introduced as a perturbation on the total motion due to all the interactions. It results in closed equations which fix the covariance matrix of the Fourier modes and a matrix, basic to the perturbation treatment, describing the impulse response of the distribution to small disturbances.

The statistical mechanical method used involves a distribution functional of time functions satisfying the equations of motion. It is, in this respect, more general than the Gibbs statistical mechanics which treats a distribution only of simultaneous amplitudes, and which, as we shall see, is inadequate for the stationary turbulence problem. Apart from its specific usefulness in the turbulence problem, the method presented is of interest because its applicability is not limited by the requirement that the state of the system be close to a state of detailed balance.

The ambiguity in the distribution functional mentioned in the Introduction is resolved by the requirement that the distribution of the Fourier modes be as independent as the equations of motion permit. This is expressed first implicitly by a hypothesis of weak statistical dependence of the modes (in a sense specifically defined) and later explicitly by a variational method of determining the higher moments of the distribution functional.

3. Representation of the fields by linearly independent variables

The equations of motion of an incompressible, highly conducting, and nonrelativistic hydromagnetic fluid subjected to external body or boundary forces may be written[5]

$$(3.1)\qquad \dot{u}_i(\underset{\sim}{k}) + \nu k^2 u_i(\underset{\sim}{k}) = ik_\ell P_{ij}(\underset{\sim}{k}) \sum_{\underset{\sim}{k}'} \left[w_j(\underset{\sim}{k}-\underset{\sim}{k}')w_\ell(\underset{\sim}{k}') - u_j(\underset{\sim}{k}-\underset{\sim}{k}')u_\ell(\underset{\sim}{k}')\right]$$

$$+ P_{ij}(\underset{\sim}{k})\, F_j(\underset{\sim}{k}),$$

$$(3.2)\qquad \dot{w}_i(\underset{\sim}{k}) + \bar{\nu} k^2 w_i(\underset{\sim}{k}) = ik_j \sum_{\underset{\sim}{k}'} \left[u_i(\underset{\sim}{k}-\underset{\sim}{k}')w_j(\underset{\sim}{k}') - u_j(\underset{\sim}{k}-\underset{\sim}{k}')w_i(\underset{\sim}{k}')\right].$$

Here $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$, $\underset{\sim}{F}(\underset{\sim}{k})$ are related to the velocity field $\hat{\underset{\sim}{u}}(\underset{\sim}{x})$, magnetic induction $\hat{\underset{\sim}{B}}(\underset{\sim}{x})$, and external force field $\hat{\underset{\sim}{F}}(\underset{\sim}{x})$ by

$$(3.3)\qquad \hat{\underset{\sim}{u}}(\underset{\sim}{x}) = \sum_{\underset{\sim}{k}} \underset{\sim}{u}(\underset{\sim}{k}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}, \quad \hat{\underset{\sim}{B}}(\underset{\sim}{x}) = (4\pi\mu\rho)^{1/2} \sum_{\underset{\sim}{k}} \underset{\sim}{w}(\underset{\sim}{k}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}, \quad \hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \rho \sum_{\underset{\sim}{k}} \underset{\sim}{F}(\underset{\sim}{k}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}},$$

and $P_{ij}(\underset{\sim}{k}) = \delta_{ij} - k^{-2} k_i k_j$. The summations range over all wave vectors allowed by the boundary conditions. The quantities ρ, ν, μ, σ, c and $\bar{\nu} = (4\pi\mu\sigma)^{-1} c^2$ denote, respectively, density, kinematic viscosity, permeability, conductivity, velocity of light *in vacuo*, and 'ohmic viscosity'; they all are assumed constant throughout the fluid. The fields also obey the divergence conditions

$$(3.4)\qquad k_i u_i(\underset{\sim}{k}) = 0, \qquad k_i w_i(\underset{\sim}{k}) = 0,$$

which are preserved by (3.1), (3.2), if imposed initially.

The total energy per unit mass is

$$(3.5)\qquad E = \tfrac{1}{2} \sum_{\underset{\sim}{k}} \left[u_i^*(\underset{\sim}{k})\, u_i(\underset{\sim}{k}) + w_i^*(\underset{\sim}{k})\, w_i(\underset{\sim}{k})\right].$$

Energy is conserved by the nonlinear interaction of the Fourier modes in the sense that if ν, $\bar{\nu}$ and $\underset{\sim}{F}(\underset{\sim}{k})$ are put equal to zero, (3.1), (3.2), (3.4) yield $\dot{E} = 0$.

We shall assume that the velocity and magnetic fields obey cyclic boundary conditions on the faces of a cube of side L. These conditions are more natural for the description of homogeneous statistical distributions than the requirement that the fields vanish on the boundaries, and the small-scale dynamics should be insensitive to the precise form of the boundary conditions. The allowed $\underset{\sim}{k}$-vectors now are all those whose components along the coordinate axes are integer multiples of $2\pi/L$. We shall further assume, however, that there is no uniform velocity or magnetic field, so that $\underset{\sim}{u}(0) = \underset{\sim}{w}(0) = 0$. These conditions are preserved by the equations of motion, provided $\underset{\sim}{F}(0) = 0$.

The coefficients $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$ for allowed $\underset{\sim}{k}$ are not linearly independent; assignment of their initial values is restricted by (3.4) and the reality requirements

$$(3.6) \qquad u_i(-\underset{\sim}{k}) = u_i^*(\underset{\sim}{k}), \qquad w_i(-\underset{\sim}{k}) = w_i^*(\underset{\sim}{k}).$$

In the statistical mechanical treatment it is a great advantage to work with variables for which an arbitrary assignment of initial values corresponds to admissible fields. For this purpose we introduce those real and imaginary parts of the vector components of the $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$ which are linearly independent under (3.4), (3.6).

The vector components of $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{u}(-\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$, $\underset{\sim}{w}(-\underset{\sim}{k})$ comprise, for given $\underset{\sim}{k}$, a total of twenty-four real and imaginary parts, of which only eight are linearly independent. We shall pick from each wave vector pair $\underset{\sim}{k}$, $-\underset{\sim}{k}$ one member, say $\underset{\sim}{k}$, and choose as these eight variables the real and imaginary parts of the components of the vectors $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$ along pairs of perpendicular axes in the plane normal to $\underset{\sim}{k}$. Let all the independent real and imaginary parts so chosen (for all pairs $\underset{\sim}{k}$, $-\underset{\sim}{k}$ allowed by the boundary conditions) be ordered in some single one-dimensional sequence and denoted by $q_\alpha/\sqrt{2}$, where $\alpha = 1,2,3,4,5, \ldots$ is a serial index whose values label individual variables. Then for given $\underset{\sim}{k}$ the realizable values of $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{u}(-\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$, $\underset{\sim}{w}(-\underset{\sim}{k})$ are fixed completely by the values of eight of the q_α. The q_α are analogous to the canonical coordinates and momenta of Hamiltonian systems, as we shall see.

It follows from the linear relations between the q's and the $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{w}(\underset{\sim}{k})$ that (3.1), (3.2) may be written in the combined form

$$(3.7) \qquad \dot{q}_\alpha + \nu_\alpha q_\alpha = \sum_{\beta,\gamma} A_{\alpha\beta\gamma}\, q_\beta\, q_\gamma + f_\alpha, \qquad (A_{\alpha\beta\gamma} = A_{\alpha\gamma\beta}),$$

where the coefficients $A_{\alpha\beta\gamma}$ and damping factors ν_α are functions of their indices (but not of the q's or of time), and the f's are related to $P_{ij}(\underset{\sim}{k})F_j(\underset{\sim}{k})$ as the q's are to $u_i(\underset{\sim}{k})$, $w_i(\underset{\sim}{k})$. (We shall frequently denote the row vectors with elements q_α and f_α by $\underset{\sim}{q}$, $\underset{\sim}{f}$. No summation convention will be used for the Greek indices.) It is readily verified from (3.1), (3.2), and the definition of the q's, that each ν_α is positive and proportional to either ν or $\bar{\nu}$.

It follows from (3.5) and the definition of the q's that

$$(3.8) \qquad E = \frac{1}{2}\sum_{\alpha} q_\alpha{}^2 .$$

We remarked earlier that if ν, $\bar{\nu}$ and $\underset{\sim}{F}(\underset{\sim}{k})$ are put equal to zero then (3.1),(3.2), (3.4) yield $\dot{E} = 0$. Therefore, setting the ν_α and f_α equal to zero in (3.7), we find

$$(3.9) \qquad \dot{E} = \sum_{\alpha} q_\alpha \dot{q}_\alpha = \sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma}\, q_\alpha\, q_\beta\, q_\gamma = 0, \qquad (\nu_\alpha,\ f_\alpha = 0).$$

Since any assignment of initial values to $\underset{\sim}{q}$ represents admissible velocity and magnetic fields, the $A_{\alpha\beta\gamma}$ must be such that (3.9) holds for all $\underset{\sim}{q}$. Setting all the q's equal to zero except $q_\mu, q_\lambda, q_\sigma$, we find

$$(3.10) \qquad A_{\mu\lambda\sigma} + A_{\lambda\sigma\mu} + A_{\sigma\mu\lambda} = 0.$$

If the <u>triangle interaction</u> of the modes $q_\mu, q_\lambda, q_\sigma$ is defined by the terms $2A_{\mu\lambda\sigma}\, q_\lambda\, q_\sigma$, $2A_{\lambda\sigma\mu} q_\sigma\, q_\mu$, $2A_{\sigma\mu\lambda} q_\mu\, q_\lambda$ in the respective equations of motion for $q_\mu, q_\lambda, q_\sigma$, it follows immediately from (3.10) that this elementary interaction makes a vanishing contribution to $\dot{E}$ in (3.9) and therefore is individually conservative.*

*The 'rate of energy-transfer from wave number $\underset{\sim}{k}'$ to wave number $\underset{\sim}{k}$ ' frequently is defined as $Q(\underset{\sim}{k},\underset{\sim}{k}') = \frac{1}{2} i \left[\underset{\sim}{k}\cdot\underset{\sim}{u}^*(\underset{\sim}{k}-\underset{\sim}{k}')\underset{\sim}{u}^*(\underset{\sim}{k}')\cdot\underset{\sim}{u}(\underset{\sim}{k}) - \underset{\sim}{k}\cdot\underset{\sim}{u}(\underset{\sim}{k}-\underset{\sim}{k}')\underset{\sim}{u}(\underset{\sim}{k}')\cdot\underset{\sim}{u}^*(\underset{\sim}{k})\right]$, since, $(\nu,\underset{\sim}{F}=0)$, $(\partial/\partial t)\left[\frac{1}{2}\,\underset{\sim}{u}^*(\underset{\sim}{k})\cdot\underset{\sim}{u}(\underset{\sim}{k})\right] = \sum_{\underset{\sim}{k}'} Q(\underset{\sim}{k},\underset{\sim}{k}')$ [cf. ref. 4]. The conservation property is here expressed by $Q(\underset{\sim}{k},\underset{\sim}{k}') + Q(\underset{\sim}{k}',\underset{\sim}{k}) = 0$. Actually, the representation of the complex of triangle interactions as a complex of conservative pair interactions is not well-defined. There is a complementary term $Q(\underset{\sim}{k},\underset{\sim}{k}-\underset{\sim}{k}')$ for each term $Q(\underset{\sim}{k},\underset{\sim}{k}')$ in $\sum_{\underset{\sim}{k}'} Q(\underset{\sim}{k},\underset{\sim}{k}')$. These two terms each depend on all three amplitudes $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{u}(\underset{\sim}{k}')$, $\underset{\sim}{u}(\underset{\sim}{k}-\underset{\sim}{k}')$ (or their complex conjugates), and it is rather capricious to call $Q(\underset{\sim}{k},\underset{\sim}{k}')$ the rate of transfer from $\underset{\sim}{k}'$ and $Q(\underset{\sim}{k},\underset{\sim}{k}-\underset{\sim}{k}')$ that from $\underset{\sim}{k}-\underset{\sim}{k}'$. This is exhibited by noting that $\bar{Q}(\underset{\sim}{k},\underset{\sim}{k}') = Q(\underset{\sim}{k},\underset{\sim}{k}') + K(k,k',|\underset{\sim}{k}-\underset{\sim}{k}'|)$, where K is any function antisymmetric in each pair of its arguments, may validly replace $Q(\underset{\sim}{k},\underset{\sim}{k}')$, since $\bar{Q}(\underset{\sim}{k},\underset{\sim}{k}') + \bar{Q}(\underset{\sim}{k}',\underset{\sim}{k}) = 0$, and $\bar{Q}(\underset{\sim}{k},\underset{\sim}{k}') + \bar{Q}(\underset{\sim}{k},\underset{\sim}{k}-\underset{\sim}{k}') = Q(\underset{\sim}{k},\underset{\sim}{k}') + Q(\underset{\sim}{k},\underset{\sim}{k}-\underset{\sim}{k}')$. It may be noted, parenthetically, that (3.10) is a stronger statement than $Q(\underset{\sim}{k},\underset{\sim}{k}') + Q(\underset{\sim}{k}',\underset{\sim}{k}) = 0$, since it does not involve averaging over the vector-components of $\underset{\sim}{u}(\underset{\sim}{k})$.

Most of the $A_{\alpha\beta\gamma}$ defined by (3.7) vanish because of properties of (3.1), (3.2). We first note, from (3.4) and our assumption $\underset{\sim}{u}(0) = \underset{\sim}{w}(0) = 0$, that only pairs of Fourier coefficients such that $\underset{\sim}{k}'$ and $\underset{\sim}{k}-\underset{\sim}{k}'$ are neither parallel nor antiparallel to $\underset{\sim}{k}$ (nor, hence, to each other) can contribute to the right sides of (3.1), (3.2). The immediate consequence is that

(3.11) $\quad A_{\alpha\beta\gamma} = 0, \quad$ unless α,β,γ are all different.

We may also see from (3.1),(3.2) that the triangle interactions connect q's representing real and imaginary parts of Fourier coefficients in only certain ways. If an even number of the three indices α,β,γ refer to imaginary part q's then $A_{\alpha\beta\gamma}, A_{\beta\gamma\alpha}, A_{\gamma\alpha\beta}$ all vanish. The coefficients describing the non-vanishing triangle interactions obey the relations

(3.12) $$A_{\alpha_r\beta_r\gamma_i} = A_{\alpha_r\beta_i\gamma_r} = -A_{\alpha_i\beta_r\gamma_r} = A_{\alpha_i\beta_i\gamma_i}$$

Here q_{α_r} and q_{α_i} are, respectively, the real and imaginary parts of a single independent vector component of some given Fourier coefficient, with a similar interpretation for the other indices.

Now let us arrange all the allowed wave vectors in some one-dimensional sequence $\underset{\sim}{k}_{(1)}, \underset{\sim}{k}_{(2)}, \ldots$. Since the summations on the right sides of (3.1),(3.2) extend over the single vector $\underset{\sim}{k}'$, the number of $A_{\alpha\beta\gamma}$ which have some single fixed index α, β or γ, and which connect modes with wave vectors below a member $\underset{\sim}{k}_{(N)}$ of the sequence, is proportional to N as $N \to \infty$. The number of nonvanishing $A_{\alpha\beta\gamma}$ with two fixed indices (implying fixed values of some pair, and hence all three, of $\underset{\sim}{k}$, $\underset{\sim}{k}'$, and $\underset{\sim}{k}-\underset{\sim}{k}'$) is a small number independent of N in the limit.

4. Formulation of the statistical problem

4.1 Gibbs ensembles. The statistical mechanics of the dynamical system obeying (3.7) may be formulated in terms of Gibbs ensembles in close analogy to that of Hamiltonian systems. The application of Gibbs statistical mechanics to turbulence has been discussed by Burgers[6], Onsager[7], Hopf[1], and other authors. We shall present briefly some results of this approach and then point out its inadequacy for the full statistical description of stationary turbulence.

The system described by (3.7) has an infinite number of degrees of freedom, and it is not possible to speak meaningfully of its equilibrium statistical mechanics ($\nu_\alpha = 0$, $f_\alpha = 0$). We avoid this difficulty by truncating

the system, removing from the equations of motion all q_α (and f_α) such that $\alpha > N$, where N is an arbitrarily large number. With a suitable ordering of the q's, this may be regarded as the introduction of an infinite ohmic or viscous damping for all modes with wave number higher than some limit. The necessity of this artifice is related to the fact that the Navier-Stokes equation for zero viscosity does not have nontrivial families of stationary solutions, but leads to the formation of surfaces of discontinuity. If N is taken large enough, the truncated system provides a valid representation for the viscous case, since we may assume that the excitation of sufficiently high wave numbers is negligible. In the rest of this paper we shall deal with the truncated system.

According to (3.7),(3.11) we have

$$\sum_\alpha \partial \dot{q}_\alpha / \partial q_\alpha = 0, \quad (\nu_\alpha = 0). \tag{4.1}$$

This is equivalent to the Liouville theorem obeyed by conservative Hamiltonian systems, and it shows that the motion conserves measure in a phase space with the q_α as Cartesian coordinates, so that for $\underset{\sim}{f} = 0$ the time-invariant ensembles defined by density $\rho(\underset{\sim}{q})$ are those which satisfy the equation

$$\sum_\alpha \dot{q}_\alpha \frac{\partial \rho}{\partial q_\alpha} = \sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma}\, q_\beta\, q_\gamma \frac{\partial \rho}{\partial q_\alpha} = 0.$$

By (3.8),(3.9), $\rho = \rho(E)$ is an obvious solution. More generally, the solutions are arbitrary functions of functions $\chi_i(\underset{\sim}{q})$ such that the relations $\chi_i(\underset{\sim}{q}) = \text{const.}$ are the integrals of the simultaneous system

$$\frac{dq_\alpha}{\sum_{\beta,\gamma} A_{\alpha\beta\gamma} q_\beta q_\gamma} = \frac{dq_\lambda}{\sum_{\beta,\gamma} A_{\lambda\beta\gamma} q_\beta q_\gamma}.$$

According to the theory of such systems, we may expect that there exist N-1 independent χ_i. This implies that there are N-1 constants of motion, including the energy, associated with the conservative equations of motion, with a corresponding restriction on the ergodicity of the truncated system.

For $\underset{\sim}{f} = 0$ the density $\rho(\underset{\sim}{q})$, prescribed at some instant, determines the complete structure of the distribution of solutions. The density at later times may be determined by Liouville's equation, and averages of any function of

simultaneous values of the q's may be evaluated directly by integration over ρ. Averages of functions of nonsimultaneous values, such as $< q_\alpha(t) q_\beta(t') >$, also are fixed by ρ, but their evaluation is not so direct. For the example given, $q_\beta(t')$ could be expressed as a Taylor series about time t, the various derivatives in the series expressed in terms of $\underset{\sim}{q}(t)$ by repeated use of the equations of motion (and their time-derivatives), and the result integrated over ρ.

Since nonsimultaneous averages are of great interest in turbulence, the lack of symmetry of the Gibbs method in this respect is a disadvantage. Moreover, when external forces act, the Gibbs method no longer determines nonsimultaneous averages. To see this, we note that when $\underset{\sim}{f} \neq 0$, (3.7) and its derivatives still may be used to express higher derivatives of $\underset{\sim}{q}$ in terms of lower, but this procedure introduces time-derivatives of $\underset{\sim}{f}$. In this case, the evaluation of $< q_\alpha(t)\ q_\beta(t') >$ by the Taylor series method requires knowledge not only of the distribution of simultaneous values of $\underset{\sim}{q}$, but also of the full simultaneous joint-distribution of these values with those of $\underset{\sim}{f}$ and all time derivatives of $\underset{\sim}{f}$. This points up the desirability of employing a more complete statistical description than is provided by the Gibbs ensemble, since in stationary turbulence external forces are a necessity, as we have formulated the problem.

4.2 Time-function distributions. The full solution of the statistical problem evidently requires the determination of a physically appropriate distribution functional $\mathbf{Y}[\underset{\sim}{q}(\ ), \underset{\sim}{f}(\ )]$ of vectors $\underset{\sim}{q}(\ )$, $\underset{\sim}{f}(\ )$ which obey (3.7).* Since $\mathbf{Y}$ describes the entire history of the systems comprising the distribution, it does not obey an equation of motion like the Liouville equation for the time-change of the Gibbs density. However, a functional equation for $\mathbf{Y}$ can be constructed which expresses necessary and sufficient conditions that the members of the distribution obey (3.7). If we make the definition

$$(4.2) \qquad L_\alpha(t) = \dot{q}_\alpha(t) + \nu_\alpha q_\alpha(t) - \sum_{\beta,\gamma} A_{\alpha\beta\gamma}\ q_\beta(t)\ q_\gamma(t) - f_\alpha(t),$$

this equation may be stated as

$$(4.3) \qquad < \sum_\alpha L_\alpha(t)\ L_\alpha(t') >_{\mathbf{Y}} = 0,$$

where $< >_{\mathbf{Y}}$ indicates an average over $\mathbf{Y}[\underset{\sim}{q}(\ ), \underset{\sim}{f}(\ )]$. The necessity of (4.3) is evident, and it is sufficient if imposed for t=t'. This follows from the fact that the left side is a sum with positive measure over a sum of squares of real

*At this stage we do not regard the distribution of external forces as necessarily prescribed a priori. The restrictions on $\mathbf{Y}$ appropriate to a particular physical situation may not best be expressible as conditions on the f's alone.

quantities, and can vanish only if each of the squares vanishes, except for a set of zero measure. The possible inclusion in the distribution of a zero-measure set of nonsolutions is of no physical significance, for they cannot affect finite-order moments. The integral of (4.3) (with $t = t'$) over an arbitrary time interval is the condition that the members of the distribution obey the equations of motion in a mean square sense within the integration interval.

Since $L_\alpha(t)$ contains only linear and bilinear terms in the variables, it is clear that (4.3) involves only second-, third-, and fourth-order moments of the distribution of $\underset{\sim}{q}$ and $\underset{\sim}{f}$. Therefore we need evaluate only these moments to determine whether a given distribution satisfies the statistical equations of all orders obtained by multiplying (3.7) with various functions of $\underset{\sim}{q}$ and $\underset{\sim}{f}$ and averaging. This suggests a high degree of indetermination of $\mathbf{Y}$ by the equations of motion, and, as indicated before, we should expect as a minimum requirement for determination of the distribution something equivalent to complete specification of the distribution of $\underset{\sim}{f}$, together with additional conditions, such as stationarity, or specification of the distribution of initial values of $\underset{\sim}{q}$.

It should be noted, however, that not every assignment of values to second-, third-, and fourth-order moments is realizable. The requirement that $\mathbf{Y}$ have everywhere non-negative measure in the function space of $\underset{\sim}{q}$ and $\underset{\sim}{f}$ implies that the distribution average of any everywhere non-negative functional of these vectors must itself be non-negative[8]. These conditions severely limit the realizable moment values.

The functional $\mathbf{Y}$ is difficult to express explicitly because it depends on a nondenumerably infinite set of variables, $\underset{\sim}{q}(t)$ and $\underset{\sim}{f}(t)$ for all t. We may avoid this difficulty by confining attention to an arbitrarily large finite time interval. Then we may expand the time-functions within this interval in Fourier series with coefficients $\underset{\sim}{\tilde{q}}(\omega)$, $\underset{\sim}{\tilde{f}}(\omega)$ and work with the corresponding functional $\mathbf{\Phi}[\underset{\sim}{\tilde{q}}(\ ),\underset{\sim}{\tilde{f}}(\ )]$. The condition on $\mathbf{\Phi}$ equivalent to (4.3) is

$$(4.4) \qquad \Big\langle \sum_{\omega} \sum_{\alpha} \tilde{L}_\alpha^*(\omega)\, \tilde{L}_\alpha(\omega) \Big\rangle_{\mathbf{\Phi}} = 0,$$

where $\tilde{L}_\alpha(\omega)$ is the Fourier series transform of $L_\alpha(t)$.

We may note that a formal solution of (4.4) is

$$\mathbf{\Phi} = \prod_{\alpha,\omega} N(\alpha,\omega)\, \delta\,(\tilde{L}_\alpha(\omega))\, \mathbf{\Phi}_{\underset{\sim}{\tilde{f}}},$$

where $\mathfrak{F}_{\tilde{f}}$ is a functional of $\tilde{f}(\ )$ only, and the N's are normalization factors. One might attempt to work from this expression by employing representations of the δ-functions, but we shall not take this approach in the present paper.

4.3 Stationarity and homogeneity. A stationary distribution is one whose moments are all invariant under time displacement. That is to say, its moments depending on quantities with arguments t_1, t_2, ..., t_n actually depend only on the n - 1 independent differences of these times. We shall assume that distribution averages over a stationary distribution can be identified with long-time averages for a representative physical system, macroscopically isolated in the dissipationless case, or subjected to stationary driving forces in the dissipative case. We may reconcile this ergodic principle with the previously noted lack of ergodicity by appealing to random perturbations which we presume destroy the constants of motion.

In a stationary distribution only the following moments, and their τ-derivatives, appear in (4.3):

$$R_{\alpha\beta}(\tau) = \langle q_\alpha(t+\tau)\, q_\beta(t) \rangle,$$

$$S_{\beta\gamma\alpha}(\tau) = \langle q_\beta(t+\tau)\, q_\gamma(t+\tau)\, q_\alpha(t) \rangle,$$

$$T_{\mu\lambda\beta\gamma}(\tau) = \langle q_\mu(t+\tau)\, q_\lambda(t+\tau)\, q_\beta(t)\, q_\gamma(t) \rangle,$$

$$F_{\alpha\beta}(\tau) = \langle f_\alpha(t+\tau)\, f_\beta(t) \rangle,$$

$$G_{\alpha\beta}(\tau) = \langle f_\alpha(t+\tau)\, q_\beta(t) \rangle,$$

$$H_{\alpha\beta\gamma}(\tau) = \langle f_\alpha(t+\tau)\, q_\beta(t)\, q_\gamma(t) \rangle. \tag{4.5}$$

We shall denote these moment matrices, with indices suppressed, by $\overline{\overline{R}}$, $\overline{\overline{S}}$, $\overline{\overline{T}}$, $\overline{\overline{F}}$, $\overline{\overline{G}}$, $\overline{\overline{H}}$. From the definitions, and stationarity, it follows that $R_{\alpha\beta}(\tau) = R_{\beta\alpha}(-\tau)$, with similar relations for the other moments. In terms of these moments (4.3) has the form

$$\sum_\alpha \mathrm{Sym} \left\{ -\dot{R}_{\alpha\alpha}(\tau) + \nu_\alpha^2 R_{\alpha\alpha}(\tau) + 2 \sum_{\beta,\gamma} A_{\alpha\beta\gamma} \left[\dot{S}_{\beta\gamma\alpha}(\tau) - \nu_\alpha S_{\beta\gamma\alpha}(\tau) + H_{\alpha\beta\gamma}(\tau) \right] \right.$$
$$\left. + \sum_{\beta,\gamma,\mu,\lambda} A_{\alpha\beta\gamma}\, A_{\alpha\mu\lambda} T_{\beta\gamma\mu\lambda}(\tau) + 2\dot{G}_{\alpha\alpha}(\tau) - 2\nu_\alpha G_{\alpha\alpha}(\tau) + F_{\alpha\alpha}(\tau) \right\} = 0, \tag{4.6}$$

where Sym $\{z(\tau) = (1/2)[z(\tau) + z(-\tau)]$, for any $z(\tau)$. The corresponding expression for (4.4) involves the Fourier series transforms of the moments in (4.6) (defined in an arbitrarily large time interval).

Homogeneous distributions are those whose moments all are invariant under spatial displacement. Under the cyclic boundary conditions we have adopted, homogeneity requires that Fourier coefficients associated with different wave number pairs $\underset{\sim}{k}$, $-\underset{\sim}{k}$ and $\underset{\sim}{k}'$, $-\underset{\sim}{k}'$ be uncorrelated. If the distribution also is invariant under spatial inversion, the real part of a given Fourier coefficient must be uncorrelated with the imaginary part. Under the stated conditions, it is possible to choose orthogonal principal axes, in the plane normal to $\underset{\sim}{k}$, along which the vector-components of $\underset{\sim}{u}(\underset{\sim}{k})$ are uncorrelated, and the same may be done for $\underset{\sim}{w}(\underset{\sim}{k})$ and $\underset{\sim}{F}(\underset{\sim}{k})$. (The choice of axes may have to be different in each case.) Our definition of $\underset{\sim}{q}$ and $\underset{\sim}{f}$ leaves us free to take their elements as components along any orthogonal axes in the normal planes; let us choose them along the principal axes just defined. Now we note, following Chandrasekhar[9], that (3.1), (3.2) are invariant under change of sign of $\underset{\sim}{w}$, so that there should exist distributions in which $\underset{\sim}{u}(\underset{\sim}{k})$ and $\underset{\sim}{w}(\underset{\sim}{k})$ are uncorrelated. With our choice of axes, it follows that with all the conditions stated,

$$(4.7) \qquad R_{\alpha\beta}(\tau) = \delta_{\alpha\beta} R_{\alpha\alpha}(\tau), \qquad F_{\alpha\beta}(\tau) = \delta_{\alpha\beta} F_{\alpha\alpha}(\tau).$$

In the rest of the present paper we shall be concerned with distributions obeying (4.7), and in which $\underset{\sim}{q}$ and $\underset{\sim}{f}$ have zero means. For their existence, we appeal to the invariance of (3.1),(3.2) under spacetime displacement, spatial inversion, and change of sign of $\underset{\sim}{w}(\underset{\sim}{k})$. The assumption of homogeneous external driving forces is physically unrealistic, as discussed further in Section 10. This assumption clearly is required, however, in any consistent idealization of turbulent motion as a rigorously homogeneous and stationary phenomenon.

4.4 The hypothesis of weak dependence. Let us suppose that the forces f_α, which maintain a state of stationary, homogeneous turbulence, are statistically independent, or nearly so. We hypothesize that in turbulence so maintained the statistical dependence among any group of n modes is weak if n is small compared to the total number of modes significantly excited.* More precisely, we assume

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*The order of magnitude of the total number of significantly excited modes may be estimated very crudely as the cube of the ratio of macroscale to microscale. It is quite large even for moderate Reynolds numbers.

that normalized moments formed from the q_α in such a group are closely approximated by the values they would have in an independent distribution, if the latter values do not vanish, and are small compared to unity if the corresponding values in an independent distribution do vanish. This hypothesis is based on the fact that the motion of each q_α is determined by coupling to f_α and by coupling, in an individual, complicated fashion, to many other excited modes, so that strong dependencies among any small group of modes would be difficult to account for. The further justification of our hypothesis must lie in the consistency of the results we shall obtain by its use, and, more logically, in constructions, or existence proofs, for distributions obeying it, as will be discussed in Section 8.

It should be noted carefully that we may not assume the weak cross-dependencies to be zero; there are very many of them, and together they represent the entire effect of the nonlinear interaction. In particular, we may not assume that the cumulants associated with fourth-order cross-moments of the q's vanish. This assumption, which has been used by Heisenberg[10], Chandrasekhar[11], and others, is seriously incompatible with the equations of motion[12].

We shall assume in the remainder of this paper that distributions appropriate to the description of stationary, homogeneous turbulence exhibit weak dependence as we have defined it.

5. The Perturbation treatment

5.1 The perturbation hypothesis. Let us consider a modification of (3.7) in which the triangle interaction of q_μ, q_λ, q_σ is removed, the coefficients $A_{\mu\lambda\sigma}$, $A_{\lambda\sigma\mu}$, $A_{\sigma\mu\lambda}$ (originally not all zero) being placed equal to zero, with the other A's unchanged. Suppose we seek, according to some suitable criterion*, the most nearly independent distribution of the q_α and f_α consistent with the modified equations of motion, and with given $F_{\alpha\alpha}(\tau)$ (or given E for non-dissipative equilibrium, ν_α, $f_\alpha = 0$). The interaction we have removed represents a small part of the total interaction, if many modes are excited; therefore, the most nearly independent distribution should not differ very much for the original and modified problems. However, we note that the moments $S_{\mu\lambda\sigma}(\tau)$, $S_{\lambda\sigma\mu}(\tau)$, $S_{\sigma\mu\lambda}(\tau)$ do not appear at all in (4.6) for the modified problem. Since this equation is a sufficient condition for $\mathbf{Y}$ to satisfy the equations of motion, our immediate surmise might be that

*Such a criterion will be presented in Section 8.

in the most independent distribution possible these moments must certainly be very small, or zero, since their values cannot affect (4.6).

Matters are not this simple, because, if other cross-moments do not vanish, zero values of $S_{\mu\lambda\sigma}(\tau)$, $S_{\lambda\sigma\mu}(\tau)$, $S_{\sigma\mu\lambda}(\tau)$ may not be realizable in a distribution with non-negative measure. On the basis of the weak dependence hypothesis it seems plausible to assume, however, that the effect of the realizability conditions in a suitably defined most nearly independent distribution may be to induce values for $S_{\mu\lambda\sigma}(\tau)$, $S_{\lambda\sigma\mu}(\tau)$, $S_{\sigma\mu\lambda}(\tau)$ at most of small magnitude compared to their values in the original problem. We therefore are led to the hypothesis that in the original problem $S_{\mu\lambda\sigma}(\tau)$, $S_{\lambda\sigma\mu}(\tau)$, $S_{\sigma\mu\lambda}(\tau)$ are well approximated by the contribution to their values obtained by introducing the triangle interaction with coefficients $A_{\mu\lambda\sigma}$, $A_{\lambda\sigma\mu}$, $A_{\sigma\mu\lambda}$ as a perturbation on the motion due to the rest of the interactions.

The argument given may be extended to $\overline{\overline{H}}(\tau)$, and those elements of $\overline{\overline{T}}(\tau)$ which vanish in a completely independent distribution of $\underset{\sim}{q}$. We notice that $H_{\alpha\beta\gamma}(\tau)$ appears in (4.6) multiplied by $A_{\alpha\beta\gamma}$; therefore we hypothesize that it is well approximated by the contribution to its value obtained by introducing as simultaneous perturbations on the motion the triangle interaction with coefficients $A_{\alpha\beta\gamma}$, $A_{\beta\gamma\alpha}$, $A_{\gamma\alpha\beta}$ and the force component f_α. (If f_α is removed from (3.7), $H_{\alpha\beta\gamma}(\tau) = 0$). Similarly, we notice that the elements of $\overline{\overline{T}}(\tau)$ with indices μ,λ,β,γ in some order appear in (4.6) multiplied bilinearly with A's involving these indices. If μ,λ,β,γ are equal in pairs, these elements of $\overline{\overline{T}}(\tau)$ are well approximated by their values in an independent distribution, according to the weak dependence hypothesis. Following the argument above, we hypothesize that for μ,λ,β,γ not equal in pairs, an element $T_{\mu\lambda\beta\gamma}(\tau)$ is well approximated by the contribution to its value obtained by introducing in pairs, as simultaneous perturbations on the motion, those triangle interactions whose coefficients bilinearly multiply $T_{\mu\lambda\beta\gamma}(\tau)$ and the elements obtained from it by index permutation.

We complete this statement of the perturbation hypothesis by noting the evident fact that the element $G_{\alpha\alpha}(\tau)$ is given by the contribution to its value obtained by introducing f_α as a perturbation on the motion.

The perturbation hypothesis just stated has a fairly clear intuitive meaning. Consider the triple correlation of q_μ, q_λ, q_σ, described, in part, by $S_{\mu\lambda\sigma}(\tau)$, $S_{\lambda\sigma\mu}(\tau)$, $S_{\sigma\mu\lambda}(\tau)$. These three modes interact both <u>directly</u>, as a result of the terms $2A_{\mu\lambda\sigma}q_\mu q_\sigma$, $2A_{\lambda\sigma\mu}q_\sigma q_\mu$, $2A_{\sigma\mu\lambda}q_\mu q_\lambda$ in their respective

equations of motion, and indirectly, through chains of other modes connected by the equations of motion. Our hypothesis implies that the triple correlation is entirely, or almost entirely, the result of the direct interaction, the small contributions of the individual chains of indirect interaction combining to form a small or zero resultant. This certainly seems plausible if we picture turbulence as a mixing process in which phase information is degraded and destroyed by the motion as a whole. In similar fashion, it may be seen in the case of $\overset{\approx}{\overset{\approx}{H}}$ and $\overset{\approx}{\overset{\approx}{T}}$ also that our hypothesis picks out the contributions of those triangle interactions which constitute the most direct paths of interaction of the modes involved.

At the end of this section we shall give a posteriori support to the consistency of the perturbation hypothesis by actually comparing direct and indirect contributions to the moments. A thoroughgoing proof of validity would depend on the construction of distribution functionals obeying the hypothesis, as discussed in Section 8.

The moments defined in (4.5) all depend on values at only two times. The arguments given above can be restated for more general moments, and we therefore extend the perturbation hypothesis to third - order moments depending on three times, and fourth-order moments depending on four times.

5.2 Perturbation expressions for skew moments. We shall call skew moments those cross-moments which must vanish in a completely independent distribution of the q_α and f_α. In order to apply the perturbation hypothesis to the evaluation of the skew moments in (4.6) we first consider the effect on the motion of $\underset{\sim}{q}$ produced by introducing at $t = 0$ an arbitrary perturbation $\underset{\sim}{\xi}(t)$ on the right side of (3.7). If $\underset{\sim}{\xi}$ is not too large, and t is not too large, in a sense to be discussed later, the response can be written

$$(5.1)\quad \delta q_\mu(t) = \sum_\lambda \int_0^t \zeta_{\mu\lambda}(t,s)\xi_\lambda(s)ds + \sum_{\lambda,\sigma} \int_\sigma^t \int_0^t \eta_{\mu\lambda\sigma}(t,s,r)\xi_\lambda(s)\xi_\sigma(r)dsdr + \ldots,$$

where

$$\zeta_{\mu\lambda}(t,s) = 0, \quad (t < s), \quad \eta_{\mu\lambda\sigma}(t,s,r) = 0 \quad (t < s \text{ or } r), \ \ldots\ ,$$

and $\delta q_\mu(t)$ is the deviation of $q_\mu(t)$ from its unperturbed value. In general, $\zeta_{\mu\lambda}(t,s)$, $\eta_{\mu\lambda\sigma}(t,s,r)$, ... are functionals of the unperturbed $\underset{\sim}{q}$ and $\underset{\sim}{f}$ and, in addition, depend explicitly on their arguments. For a linear system, only the first term of this series is non-zero, and $\zeta_{\mu\lambda}(t,s)$ is simply the impulse response

matrix of the system. In the nonlinear case at hand, $\zeta_{\mu\lambda}(t,s)$ gives the response to an infinitesimal impulse.

Now, in order to evaluate the triple moment of q_α, q_β, q_γ by the perturbation hypothesis, we take $\xi_\alpha = 2A_{\alpha\beta\gamma}q_\beta q_\gamma$, $\xi_\beta = 2A_{\beta\gamma\alpha}q_\gamma q_\alpha$, $\xi_\gamma = 2A_{\gamma\alpha\beta}q_\alpha q_\beta$, to represent the switching on of the triangle interaction of these modes. To first order we have

$$(5.2)\qquad \langle q_\alpha(t-\tau)q_\beta(t)q_\gamma(t)\rangle \sim \langle \delta q_\alpha(t-\tau)q_\beta(t)q_\gamma(t)\rangle + \langle q_\alpha(t-\tau)\delta q_\beta(t)q_\gamma(t)\rangle + \langle q_\alpha(t-\tau)q_\beta(t)\delta q_\gamma(t)\rangle,$$

where $\delta q_\alpha(t-\tau)$, $\delta q_\beta(t)$, $\delta q_\gamma(t)$ are given by the first term in series (5.1).

The weak-dependence hypothesis implies that $\zeta_{\alpha\alpha}$ is at most weakly statistically dependent on q_β and q_γ. A further consequence of the hypothesis is that on the average, $\zeta_{\alpha\beta}$ and $\zeta_{\alpha\gamma}$ are very small compared to the diagonal element $\zeta_{\alpha\alpha}$, since $\zeta_{\alpha\beta}(t,s)$, to take one of the pair, is the response of q_α to a perturbation in the equation of motion of q_β, to which it is only weakly coupled. It follows that the first term on the right of (5.2) is well approximated by

$$(5.3)\qquad \langle \delta q_\alpha(t-\tau)q_\beta(t)q_\gamma(t)\rangle \sim 2A_{\alpha\beta\gamma}\int_0^{t-\tau}\langle \zeta_{\alpha\alpha}(t-\tau,s)\rangle\langle q_\beta(t)q_\gamma(t)q_\beta(s)q_\gamma(s)\rangle\,ds \sim 2A_{\alpha\beta\gamma}\int_0^{t-\tau}\langle \zeta_{\alpha\alpha}(t-\tau,s)\rangle R_{\beta\beta}(t-s)R_{\gamma\gamma}(t-s)\,ds,$$

with corresponding expressions for the other two terms. The averages are over the unperturbed distribution, and we have used (3.11), in addition to the weak dependence hypothesis, to obtain the third member.

Now we note that since the unperturbed distribution is stationary, $\langle \zeta_{\mu\lambda}(t,s)\rangle$ can depend only on the <u>difference</u> of its arguments, so that we may define the distribution-averaged infinitesimal impulse response matrix by

$$(5.4)\qquad g_{\mu\lambda}(t-s) = \langle \zeta_{\mu\lambda}(t-s)\rangle,$$

$$g_{\mu\lambda}(t-s) = 0, \quad (t < s).$$

(We shall refer to $\tilde{\tilde{g}}$ as simply the impulse response matrix.) We may now rewrite (5.3) as

$$(5.5) \qquad < \delta q_\alpha(t-\tau) q_\beta(t) q_\gamma(t) > \sim 2A_{\alpha\beta\gamma} \int_{-t}^{-\tau} g_{\alpha\alpha}(-\tau-s) R_{\beta\beta}(-s) R_{\gamma\gamma}(-s)\, ds.$$

We should note that, for a given perturbation strength, we cannot expect series (5.1) to converge for arbitrarily large t. In general, we should expect nonlinear response effects, represented by the second and succeeding terms of the series, to become of importance when $\delta\underset{\sim}{q}(t)$ predicted by the linear term becomes comparable with typical unperturbed values of $\underset{\sim}{q}$. At Reynolds numbers characterizing fully developed turbulence, a given triangle interaction represents only a small part of the total force acting on a given mode; therefore we may expect the error due to approximating series (5.1) by its first term to be significant only for values of t of the order many times the characteristic period of the mode whose perturbation by this interaction we are considering.

Returning to (5.5), we note that $R_{\beta\beta}(\tau)$, $R_{\gamma\gamma}(\tau)$ must fall off with increase of τ rapidly enough for $\int_0^\infty R_{\beta\beta}(t)dt$, $\int_0^\infty R_{\gamma\gamma}(\tau)d\tau$ to be finite, and we may anticipate that $g_{\alpha\alpha}(\tau)$ goes to zero at $\tau = \infty$.* It then is clear, providing τ itself is not large compared to characteristic periods, that the contribution to (5.5) from large arguments $-\tau-s$ is small, and that for large t we may replace the limit $-t$ by $-\infty$ without appreciably altering the value of the integral. The new expression depends only on the difference time τ; it therefore corresponds to the new stationary state which eventually is reached after switching on the perturbation. For τ large compared to characteristic periods, the first-order perturbation expression may involve a significant fractional error, but for such τ the triple correlation is very small in any event.

Working out for the other two terms in (5.2) the expressions corresponding to (5.5), setting $t = \infty$, and using $R_{\mu\mu}(s) = R_{\mu\mu}(-s)$, we obtain the complete first-order perturbation result:

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*This is reasonably clear, aside from other arguments, if the system is dissipative. For the nondissipative case, see Section 6.

$$(5.6)\qquad S_{\beta\gamma\alpha}(\tau) = 2A_{\alpha\beta\gamma}\int_{\tau}^{\infty} g_{\alpha\alpha}(s-\tau)R_{\beta\beta}(s)R_{\gamma\gamma}(s)ds$$

$$+ 2A_{\beta\gamma\alpha}\int_{0}^{\infty} R_{\alpha\alpha}(s-\tau)g_{\beta\beta}(s)R_{\gamma\gamma}(s)ds$$

$$+ 2A_{\gamma\alpha\beta}\int_{0}^{\infty} R_{\alpha\alpha}(s-\tau)R_{\beta\beta}(s)g_{\gamma\gamma}(s)ds.$$

Next we may evaluate $G_{\alpha\alpha}(\tau)$ by introducing f_α as a perturbation in the equation of motion of q_α. At first sight it appears that first-order perturbation theory should be inapplicable, since, at least for modes in the energy-containing range, f_α can hardly be regarded as a small perturbation. Actually, we may use the first-order theory if f_α has a normal, or nearly normal, univariate distribution. In this case, we may consider f_α the sum of a large number of small independent increments, and evaluate the covariance of q_α with each of these increments by the first-order theory. Writing $f_\alpha = \sum a_n f_{\alpha n}$, we obtain in direct analogy to the preceding analysis

$$(5.7)\qquad a_n < f_{\alpha n}(t)\delta q_\alpha(t-\tau) > \sim a_n^2\int_{\tau}^{\infty} g_{\alpha\alpha}(s-\tau)F_{\alpha\alpha nn}(s)ds, \qquad (t \to \infty),$$

where $F_{\alpha\alpha nn}(s) = < f_{\alpha n}(t+s)f_{\alpha n}(t) >$. Summing over all the $f_{\alpha n}$, we have

$$(5.8)\qquad G_{\alpha\alpha}(\tau) = \int_{\tau}^{\infty} g_{\alpha\alpha}(s-\tau)F_{\alpha\alpha}(s)ds,$$

since $F_{\alpha\alpha}(\tau) = \sum a_n^2 F_{\alpha\alpha nn}(\tau)$, by virtue of the independence of the $f_{\alpha n}$. This is precisely the result we would have obtained by considering f_α itself a small perturbation.

The evaluation of $H_{\alpha\beta\gamma}(\tau)$ and $T_{\mu\lambda\beta\gamma}(\tau)$ according to the perturbation hypothesis entails the use of second-order perturbation theory, since contributions bilinear in pairs of couplings are required. The second-order perturbation corresponding to $H_{\alpha\beta\gamma}(\tau)$ is

$$(5.9)\qquad \delta_2 < f_\alpha(t+\tau)q_\beta(t)q_\gamma(t) > = < f_\alpha(t+\tau)\delta_1 q_\beta(t)\delta_1 q_\gamma(t) >$$

$$+ < f_\alpha(t+\tau)\delta_2 q_\beta(t)q_\gamma(t) > + < f_\alpha(t+\tau)q_\beta(t)\delta_2 q_\gamma(t) >.$$

The first-order perturbations $\delta_1 q_\beta$ and $\delta_1 q_\gamma$ may be evaluated as before. The second-order perturbations $\delta_2 q_\beta$, $\delta_2 q_\gamma$ consist of contributions arising from the second term of series (5.1) and of contributions obtained by substituting for $\underset{\sim}{\xi}$ in the first term of (5.1) its first-order perturbation $\delta_1 \underset{\sim}{\xi}$. In the case at hand the first order perturbations indicated by the perturbation hypothesis are $\xi_\alpha = f_\alpha + 2A_{\alpha\beta\gamma} q_\beta q_\gamma$, $\xi_\beta = 2A_{\beta\gamma\alpha} q_\gamma q_\alpha$, $\xi_\gamma = 2A_{\gamma\alpha\beta} q_\alpha q_\beta$, so that $\delta_1 \xi_\beta = 2A_{\beta\gamma\alpha}(\delta_1 q_\gamma q_\alpha + q_\gamma \delta_1 q_\alpha)$, etc.

It is clear that the leading contributions to (5.9) from the second term in series (5.1) involve distribution averages of products of odd numbers of q's. On the basis of weak dependence, this contribution may be neglected in comparison to that arising from double application of the first order perturbation formulas, which gives expressions with an even number of q factors. By hypothesis, $H_{\alpha\beta\gamma}(\tau)$ is well approximated by those contributions arising from perturbations of the q's which involve both f_α and one of the coefficients $A_{\alpha\beta\gamma}$, $A_{\beta\gamma\alpha}$, $A_{\gamma\alpha\beta}$. This excludes the first term on the right side of (5.9). Then, evaluating $\delta_2 q_\beta(t)$ and $\delta_2 q_\gamma(t)$ as was indicated, retaining only terms not excluded by the perturbation hypothesis and the weak dependence hypothesis, we find

$$(5.10) \qquad H_{\alpha\beta\gamma}(\tau) = 2\int_0^\infty ds \int_s^\infty dr\, g_{\alpha\alpha}(r-s)\Big[A_{\beta\gamma\alpha} g_{\beta\beta}(s) R_{\gamma\gamma}(s) + A_{\gamma\alpha\beta} g_{\gamma\gamma}(s) R_{\beta\beta}(s)\Big] F_{\alpha\alpha}(r+\tau).$$

It is reassuring as to consistency that, in arriving at (5.10), the terms excluded by the perturbation hypothesis are those which are small anyhow as a consequence of weak dependence.

According to the weak dependence hypothesis, the non-skew cross-moments $T_{\beta\gamma\mu\lambda}(\tau)$ appearing in (4.6) are well approximated by expressions, in terms of $\underset{\approx}{\underset{\sim}{R}}$, corresponding to an independent distribution. The evaluation of the skew $T_{\beta\gamma\mu\lambda}(\tau)$ proceeds in a fashion very similar to that for $H_{\alpha\beta\gamma}(t)$. There are many second-order terms involved, however, and the result is complicated. It may be seen from the selection rules on the A's that for all of the skew $T_{\beta\gamma\mu\lambda}(\tau)$ except a fraction proportional to $1/N$, where N is the total number of modes in the truncated system, it is the case that if $T_{\beta\gamma\mu\lambda}(\tau)$ appears in (4.6) then there are no non-vanishing A's with one index β or γ and another index simultaneously μ or λ. With these conditions (which result in a simplification),

$$(5.11)\quad T_{\beta\gamma\mu\lambda}(\tau) = 4\sum_{\alpha} A_{\beta\gamma\alpha}A_{\mu\lambda\alpha}\int_0^\infty ds\int_\tau^\infty dr\, g_{\beta\beta}(s)R_{\gamma\gamma}(s)R_{\alpha\alpha}(r-s)g_{\mu\mu}(r-\tau)R_{\lambda\lambda}(r-\tau),$$

$$+ \text{ 3 similar terms obtained by permuting } \beta \text{ with } \gamma \text{ and/or } \mu \text{ with } \lambda,$$

$$+ 4\sum_{\alpha} A_{\beta\gamma\alpha}A_{\alpha\mu\lambda}\int_0^\infty ds\int_s^\infty dr\, g_{\beta\beta}(s)R_{\gamma\gamma}(s)g_{\alpha\alpha}(r-s)R_{\mu\mu}(r-\tau)R_{\lambda\lambda}(r-\tau),$$

$$+ \text{ a similar term obtained by permuting } \beta \text{ with } \gamma,$$

$$+ 4\sum_{\alpha} A_{\mu\lambda\alpha}A_{\alpha\beta\gamma}\int_\tau^\infty ds\int_s^\infty dr\, R_{\beta\beta}(r)R_{\gamma\gamma}(r)g_{\alpha\alpha}(r-s)g_{\mu\mu}(s-\tau)R_{\lambda\lambda}(s-\tau),$$

$$+ \text{ a similar term obtained by permuting } \mu \text{ with } \lambda,$$

$$(\beta,\gamma,\mu,\lambda \text{ all different}).$$

The moments given by (5.11) are not zero, in general. Thus, second and fourth moments are not related as in a normal distribution. The expressions for the remaining skew $T_{\beta t\mu\lambda}(\tau)$ are similar to (5.11).

We shall now present, in skeleton form, an argument in support of the a posteriori plausibility of our perturbation hypothesis for the calculation of the moments $S_{\beta\gamma\alpha}(\tau)$. Similar arguments exist for the other moments. An evident check on the consistency of the hypothesis is the comparison with (5.6) of the perturbation-theory contributions to $S_{\beta\gamma\alpha}(\tau)$ from successively longer chains of indirect interaction of q_α, q_β, q_γ, through triangle interactions with other modes. We shall indicate the essential results of such a program. First, it is not hard to verify that contributions from chains of even numbers of triangle interactions are negligible. Crudely speaking, this is because such chains lead to distribution averages over products of odd numbers of the q's. The leading terms in the contributions from chains of odd numbers of triangle interactions cancel in pairs as a result of symmetry properties of the A's. For example, it is not hard to verify that typical leading third-order contributions (involving a chain of three indirect interactions) are of the form $\sum_{\mu,\lambda,\sigma} A_{\beta\mu\lambda}A_{\lambda\gamma\sigma}A_{\sigma\mu\alpha}Z(\beta,\gamma,\alpha,\mu,\lambda,\sigma)$ where Z is a functional of the diagonal elements, with the indices indicated, of the matrices $\tilde{\tilde{g}}(\ )$ and $\tilde{\tilde{R}}(\ )$. For every term in this summation there is a conjugate term $A_{\beta\mu'\lambda'}A_{\lambda'\gamma\sigma'}A_{\sigma'\mu'\alpha}Z(\beta,\gamma,\alpha,\mu',\lambda',\sigma')$ in which one of the pair q_μ, $q_{\mu'}$ is the imaginary part of the Fourier coefficient of which the other is the real part, with similar meanings for the other primed indices. A homogeneous distribution is necessarily symmetric with respect to real and imaginary parts in the sense $g_{\mu\mu}(\tau) = g_{\mu'\mu'}(\tau)$, $R_{\mu\mu}(\tau) = R_{\mu'\mu'}(\tau)$, etc. Therefore $Z(\beta,\gamma,\alpha,\mu,\lambda,\sigma) = Z(\beta,\gamma,\alpha,\mu',\lambda',\sigma')$. However, it readily may be verified from (3.12) that for any choice of $\beta,\gamma,\alpha,\mu,\lambda,\sigma$ we have $A_{\beta\mu\lambda}A_{\lambda\gamma\sigma}A_{\sigma\mu\alpha} = -A_{\beta\mu'\lambda'}A_{\lambda'\gamma\sigma'}A_{\sigma'\mu'\alpha}$. Thus the terms in the summation cancel in pairs. The same result may be verified for

the other leading third-order contributions to $S_{\beta\gamma\alpha}(\tau)$ in which the indices β,γ,α appear differently. The order of magnitude of the individual third-order terms is such that, if they all had the same sign, the total third-order contribution would be the same order of magnitude as the direct contribution. The effect of the cancellation is to insure that the total third-order indirect contribution is negligible compared to the direct contribution (5.6), if, indeed, it does not vanish exactly. The same is true of the other higher-order contributions to $S_{\beta\gamma\alpha}(\tau)$, and of the indirect contributions to the other skew moments, to the extent that they have been investigated.

There exists, therefore, the possibility that the perturbation hypothesis is exactly true for homogeneous distributions. The more rigorous development of the argument just presented is most appropriately carried out in terms of explicit distribution functionals of the type discussed in Section 8.

6. Covariance equations for non-dissipative equilibrium

When $\nu_\alpha, f_\alpha = 0$, and the simultaneous distribution of the q_α is Maxwell-Boltzmann (M-B), a very simple exact expression may be obtained for $\tilde{\tilde{g}}(\tau)$, using only (3.8) and (4.1) and the fact that the equations of motion are of first order.

Consider two conservative systems with variables q_α and p_α which obey the equations of motion

$$(6.1) \qquad \dot{q}_\alpha = Q_\alpha(\underset{\sim}{q},t), \qquad \dot{p}_\alpha = P_\alpha(\underset{\sim}{p},t),$$

have energies

$$(6.2) \qquad E_q = \tfrac{1}{2}\sum_\alpha q_\alpha{}^2, \qquad E_p = \tfrac{1}{2}\sum_\alpha p_\alpha{}^2,$$

and obey the Liouville theorems

$$(6.3) \qquad \sum_\alpha \partial\dot{q}_\alpha/\partial q_\alpha = 0, \qquad \sum_\alpha \partial\dot{p}_\alpha/\partial p_\alpha = 0.$$

Consider M-B distributions of each of these systems with identical temperatures. The joint distribution of simultaneous amplitudes then is a function only of the total energy $E_q + E_p$, and is time-invariant.

Let us introduce a conservative coupling by altering the equations of

motion to the form

$$(6.4) \qquad \dot{q}_\alpha = Q_\alpha + \varepsilon \sum_\beta A_{\alpha\beta}(t)p_\beta, \quad \dot{p}_\alpha = P_\alpha - \varepsilon \sum_\beta A_{\beta\alpha}(t)q_\beta,$$

where $A_{\alpha\beta}(t)$ is an arbitrary matrix function of time which vanishes for $t < 0$, and ε is an infinitesimal parameter. Equation (6.3) is unaffected, and the joint simultaneous distribution is invariant under the perturbation. It follows that

$$(6.5) \qquad \delta < q_\mu(t)p_\lambda(t) > = < \delta q_\mu(t)p_\lambda(t) > + < q_\mu(t)\delta p_\lambda(t) > = 0,$$

for all μ and λ. Introducing distribution-averaged infinitesimal impulse response matrices, as in the last section, and noting the rigorous independence of the full all-time unperturbed distributions of the q-systems and p-systems, we may write (6.5) as

$$(6.6) \qquad \sum_{\alpha,\beta} \left[\int_0^t A_{\alpha\beta}(s)\left[g_{\mu\alpha}(t-s)U_{\lambda\beta}(t-s) - h_{\lambda\beta}(t-s)R_{\mu\alpha}(t-s)\right]ds = 0,\right.$$

where $\overline{\overline{h}}(\tau)$ and $\overline{\overline{U}}(\tau)$ are, respectively, the averaged infinitesimal impulse response matrix and covariance matrix of the p's. Equation (6.6) holds for arbitrary t, arbitrary $A_{\alpha\beta}(s)$, arbitrary forms of $R_{\mu\alpha}(\tau)$ and $U_{\lambda\beta}(\tau)$, and for all μ and λ. This is possible only if

$$(6.7) \qquad \frac{R_{\mu\alpha}(\tau)}{g_{\mu\alpha}(\tau)} = \frac{U_{\lambda\beta}(\tau)}{h_{\lambda\beta}(\tau)} = \text{const.}, \qquad (\tau > 0).$$

It follows from the definition of $g_{\mu\alpha}(\tau)$, and the form of (6.1), that $g_{\mu\alpha}(+0) = \delta_{\mu\alpha}$. The solution of (6.7) is then

$$(6.8) \qquad g_{\mu\alpha}(\tau) = R_{\mu\alpha}(\tau)/R_{\mu\mu}(0), \quad h_{\lambda\beta}(\tau) = U_{\lambda\beta}(\tau)/U_{\lambda\lambda}(0), \qquad (\tau > 0),$$

where we note that in the M-B distribution we have taken, $R_{\mu\alpha}(0) = \delta_{\mu\alpha}R_{\mu\mu}(0)$, $U_{\lambda\beta}(0) = \delta_{\lambda\beta}U_{\lambda\lambda}(0)$, and $R_{\mu\mu}(0) = U_{\lambda\lambda}(0)$, for all index values. In matrix notation our final result for the q system is

$$(6.9) \qquad \overline{\overline{g}}(\tau) = \overline{\overline{R}}^{-1}(0)\overline{\overline{R}}(\tau), \qquad (\tau > 0),$$

$$\overline{\overline{g}}(\tau) = 0, \qquad (\tau < 0).$$

Equation (6.9) is an exact result for an M-B distribution. This distribution seems the most appropriate physically, but it is of interest to note that for systems with very many degrees of freedom, and weak dependence, (6.9) should be an excellent approximation in a simultaneous distribution which is any reasonable function of E_q. To see this, we may consider introducing an arbitrary infinitesimal conservative coupling not with a second system, but among the q's themselves. We then obtain (6.6) with $\overset{\approx}{h}$ and $\overset{\approx}{U}$ replaced by $\overset{\approx}{g}$ and $\overset{\approx}{R}$. We then may argue that $A_{\alpha\beta}(s)$ may be chosen to give arbitrary coupling between modes whose covariances are essentially unrelated, and we are thereby led to the equivalent of (6.7), and to (6.9). We may note, also, that for a linear system the impulse response must be independent of distribution so that (6.9) gives $\overset{\approx}{g}(\tau)$ for an individual system, if $\overset{\approx}{R}(\tau)$ refers to any distribution displaying equipartition.* (The system obeying (3.7) is not linear, of course.)

If we multiply (3.7) by non-simultaneous q_α and average over Υ, we obtain

$$(6.10) \qquad \dot{R}_{\alpha\alpha}(\tau) = \sum_{\alpha} A_{\alpha\beta\gamma} S_{\beta\gamma\alpha}(\tau), \qquad (\nu_\alpha,\ f_\alpha = 0).$$

Expressing equipartition at equilibrium by $R_{\mu\mu}(0) = R$, we find, using (3.10), (5.6) and (6.9),

$$(6.11) \qquad S_{\beta\gamma\alpha}(\tau) = -\frac{2A_{\alpha\beta\gamma}}{R}\int_0^\tau R_{\alpha\alpha}(\tau-s)R_{\beta\beta}(s)R_{\gamma\gamma}(s)ds.$$

We see that $S_{\beta\gamma\alpha}(\tau)$ is an odd function of τ. ($S_{\beta\gamma\alpha}(0)$ must vanish since the Gibbs ensemble is a function only of $E = \frac{1}{2}\sum q_\alpha^2$.) We also note that $S_{\beta\gamma\alpha}(\tau)$ goes to zero as $\tau \to \infty$, provided $\overset{\approx}{R}(\tau)$ falls off properly for large argument.** By (6.10)

*For a linear system, (6.9) is anticipated by the work of H.B. Callen and his coworkers, who use, however, a different approach [13], [14]. The relation between the correlation and response matrices associated with small fluctuations from equilibrium of macroscopic thermodynamic variables is examined by Callen and Greene [15].

**An interesting estimate on the order of magnitude of the triple moment given by (6.11) may readily be obtained for an idealized system. Suppose the coupling is such that all the modes have essentially the same characteristic period, assume all the nonvanishing A's to be of the same order of magnitude and let the average number of nonvanishing A's with one fixed index be M. Then this period may be estimated by squaring (3.7) and averaging [see [12] for the method], and consequently it may be found that the maximum values attained by the normalized triple moments are of order $M^{-1/2}$. Since, with the properties given in (3.11), seq., M is the order of the total number of excited modes, this estimate supports the consistency of the first-order perturbation treatment.

and (6.11),

$$\dot{R}_{\alpha\alpha}(\tau) + 2 \sum_{\beta,\gamma} \frac{(A_{\alpha\beta\gamma})^2}{R} \int_0^\tau R_{\alpha\alpha}(\tau-s)\, R_{\beta\beta}(s)\, R_{\gamma\gamma}(s)\, ds = 0. \tag{6.12}$$

Noting that $g_{\alpha\alpha}(+0) = 1$, we may rewrite (6.12) in the 'universal' form

$$g_{\alpha\alpha}(x) = 1 - 2 \sum_{\beta,\gamma} (A_{\alpha\beta\gamma})^2 \int_0^x dy \int_0^y dz\, g_{\alpha\alpha}(y-z)\, g_{\beta\beta}(z)\, g_{\gamma\gamma}(z), \quad (x \geq 0), \tag{6.13}$$

where $x = \tau R^{1/2}$. Assuming the homogeneity conditions expressed by (4.7), we have now determined the complete covariance matrix, for (6.13) may be solved by step-by-step numerical integration, starting at $x = 0$.*

We may verify easily that if the moments we have obtained are realizable they satisfy the fundamental statistical equation (4.6). As discussed in Section 4, (4.6) is satisfied for all τ by any realizable distribution which satisfies it for $\tau = 0$. For ν_α, f_α, $\tau = 0$, (4.6) becomes

$$\sum_{\alpha} \left[-\ddot{R}_{\alpha\alpha}(0) + 2 \sum_{\beta,\gamma} A_{\alpha\beta\gamma}\, \dot{S}_{\beta\gamma\alpha}(0) + \sum_{\beta,\gamma,\mu,\lambda} A_{\alpha\beta\gamma}\, A_{\alpha\mu\lambda}\, T_{\mu\lambda\beta\gamma}(0) \right] = 0. \tag{6.14}$$

Now $T_{\mu\lambda\beta\gamma}(0)$ vanishes in a simultaneous distribution depending only on $E = \frac{1}{2} \sum q_\alpha^2$, unless the indices are equal in pairs. If we assert weak statistical dependence of the modes, we have for the latter case,

$$T_{\mu\lambda\beta\gamma}(0) \approx (\delta_{\mu\beta}\delta_{\lambda\gamma} + \delta_{\mu\gamma}\delta_{\lambda\beta} + \delta_{\mu\lambda}\delta_{\beta\gamma})R^2, \tag{6.15}$$

$(\mu,\lambda,\beta,\gamma$ not all equal),

so that, taking account of (3.11),

$$\sum_{\beta,\gamma,\mu,\lambda} A_{\alpha\beta\gamma} A_{\alpha\mu\lambda} T_{\mu\lambda\beta\gamma}(0) \approx 2 \sum_{\beta,\gamma} (A_{\alpha\beta\gamma})^2\, R^2.$$

This is an exact result if the Gibbs ensemble is M-B. By differentiation of (6.11)

*For a linear dynamic system, the equations analogous to (6.13) become a system of simultaneous bilinear algebraic equations under Laplace transformation.

(noting $\dot{R}_{\alpha\alpha}(0) = 0$) we obtain

$$(6.16) \qquad A_{\alpha\beta\gamma}\,\dot{S}_{\beta\gamma\alpha}(0) = -2\sum_{\beta,\gamma}(A_{\alpha\beta\gamma})^2 R^2.$$

From (6.16) and the derivative of (6.10) for $\tau = 0$, we obtain (6.14). It may be verified, using the perturbation expressions for $\overline{\overline{T}}(\tau)$, that (4.6) is satisfied by our moments for $\tau \neq 0$ also.

7. Covariance equations for the dissipative stationary state

Stationary turbulence, with viscosity and driving forces acting, represents a state of strong departure from equipartition and detailed balance in which the energy of the system fluctuates. In this case the method used in the last section to evaluate $\overline{\overline{g}}(\tau)$ is inapplicable. We can deduce, however, a set of integro-differential equations which simultaneously determine the $g_{\alpha\alpha}(\tau)$ and the $R_{\alpha\alpha}(\tau)$.

Multiplying (3.7) by nonsimultaneous f_α, and averaging over $\mathbf{Y}$, we obtain

$$(7.1) \qquad -\dot{G}_{\alpha\alpha}(\tau) + \nu_\alpha G_{\alpha\alpha}(\tau) = \sum_{\beta,\gamma} A_{\alpha\beta\gamma} H_{\alpha\beta\gamma}(\tau) + F_{\alpha\alpha}(\tau).$$

If now we introduce an infinitesimal perturbation δf_α, statistically independent of f_α, and with auto-covariance $\delta_2 F_{\alpha\alpha}(\tau)$, the perturbations in $G_{\alpha\alpha}(\tau)$ and $H_{\alpha\beta\gamma}(\tau)$ may be obtained immediately from (5.8),(5.10) by substituting $\delta_2 F_{\alpha\alpha}$ for $F_{\alpha\alpha}$. With some manipulation, the use of partial integration on $\delta_2 \dot{G}_{\alpha\alpha}(\tau)$ (noting $g_{\alpha\alpha}(+0) = 1$ and taking $\delta_2 F_{\alpha\alpha}(\tau)$ zero for $\tau = \infty$), we obtain the following expressions for the perturbations in the terms of (7.1):

$$(7.2) \qquad \delta_2 F_{\alpha\alpha}(\tau) \equiv \delta_2 F_{\alpha\alpha}(\tau),$$

$$\delta_2 G_{\alpha\alpha}(\tau) = \int_0^\infty g_{\alpha\alpha}(r)\,\delta_2 F_{\alpha\alpha}(r+\tau)\,dr,$$

$$-\delta_2 \dot{G}_{\alpha\alpha}(\tau) = \int_0^\infty \dot{g}_{\alpha\alpha}(r)\,\delta_2 F_{\alpha\alpha}(r+\tau)\,dr + \delta_2 F_{\alpha\alpha}(\tau),$$

$$\delta_2 H_{\alpha\beta\gamma}(\tau) = 2\int_0^\infty \int_0^\infty g_{\alpha\alpha}(r-s)\Big[A_{\beta\gamma\alpha}\, g_{\beta\beta}(s)\, R_{\gamma\gamma}(s) + A_{\gamma\alpha\beta}\, g_{\gamma\gamma}(s)\, R_{\beta\beta}(s)\Big]\,\eta(r-s)\,\delta_2 F_{\alpha\alpha}(r+\tau)\,dr\,ds,$$

where $\eta(x) = \frac{1}{2}(x+|x|)/|x|$. Equation (7.1) can hold for arbitrary $\delta_2 F_{\alpha\alpha}$ only if

the coefficients of $\delta_2 F_{\alpha\alpha}(r+\tau)$ in the integrands of its several terms total to zero. We thereby find

$$(7.3)\qquad \dot{g}_{\alpha\alpha}(\tau) + \nu_\alpha g_{\alpha\alpha}(\tau) = 2\sum_{\beta,\gamma} A_{\alpha\beta\gamma}\int_0^\tau g_{\alpha\alpha}(\tau-s)\Big[A_{\beta\gamma\alpha}g_{\beta\beta}(s)R_{\gamma\gamma}(s) + A_{\gamma\alpha\beta}g_{\gamma\gamma}(s)\,R_{\beta\beta}(s)\Big]ds, \qquad (\tau > 0).$$

Multiplying (3.7) by nonsimultaneous q_α, and averaging over Γ, we obtain

$$(7.4)\qquad \dot{R}_{\alpha\alpha}(\tau) + \nu_\alpha R_{\alpha\alpha}(\tau) = \sum_{\beta,\gamma} A_{\alpha\beta\gamma}\,S_{\beta\gamma\alpha}(\tau) + G_{\alpha\alpha}(\tau)$$

Equations (7.3), (7.4), (5.6), and (5.8) form a complete simultaneous system which, with the boundary conditions $g_{\alpha\alpha}(+0) = 1$, $R_{\alpha\alpha}(\infty) = 0$, determine both the $g_{\alpha\alpha}(t)$ and the $R_{\alpha\alpha}(\tau)$ in terms of the external force covariance $F_{\alpha\alpha}(\tau)$. To verify the consistency of these results with those of the last section, we take $\nu_\alpha = F_{\alpha\alpha} = 0$ in (7.3), (7.4), assert equipartition, and assert (6.9). Then, using (3.10), we readily find that (7.3) and (7.4) become identical with each other, and with (6.12). It may be verified that if (7.3), (7.4) are satisfied, then (4.6) is satisfied by the perturbation theory moments.

At first sight it might be thought that the system (7.3), (7.4) does not constitute a much more difficult problem than (6.12), aside from the increase in the number of unknowns by a factor of two. Actually, the difference in complexity is profound. Equations (7.3), (7.4) cannot be solved by straightforward step-by-step integration, starting with $g_{\alpha\alpha}(+0)$, $R_{\alpha\alpha}(0)$. In the first place the dependence of $R_{\alpha\alpha}(0)$ on α is one of the answers we seek, and, moreover, the right sides of (7.3), (7.4) even for small τ depend on $\tilde{\tilde{g}}$ and $\tilde{\tilde{R}}$ for all argument values. The increase in complexity is not surprising when we consider that the equilibrium nondissipative problem could have been solved in elementary fashion by expanding $R_{\alpha\alpha}(\tau)$ as a Taylor series about $\tau = 0$, expressing the derivatives in terms of simultaneous averages by repeated use of the equations of motion, and averaging over the time-invariant simultaneous distribution. No such direct procedure exists for the dissipative case.

Although in the dissipative case there may be no general relation between $g_{\alpha\alpha}(\tau)$ and $R_{\alpha\alpha}(\tau)$ of simplicity comparable to (6.9), we may conjecture that at least qualitatively correct results will be given by the assumption

$$(7.5) \qquad g_{aa}(\tau) \sim \frac{R_{aa}(\tau)}{R_{aa}(0)} e^{-\nu_a \tau}, \qquad (\tau > 0).$$

We note that with $\nu_a = 0$, (7.5) reduces to (6.9), and (since $\dot{R}_{aa}(0) = 0$) it implies $\dot{g}_{aa}(+0) + \nu_a g_{aa}(+0) = 0$, as required by (7.3). For ν_a very large, (7.5) implies $g_{aa}(\tau) \sim \exp(-\nu_a \tau)$, $(\tau > 0)$, the correct result for the impulse response of a system controlled by linear damping. This property indicates at least the qualitative correctness of (7.5) for heavily damped modes in the dissipation range of turbulence. For the inertial and low wave number ranges, where the factor $\exp(-\nu_a \tau)$ may be ignored, the qualitative correctness of (7.5) seems plausible on the intuitive ground that the time it takes a mode to forget a perturbation impulse through the mixing action of the other modes with which it interacts should be approximated by the mode's own characteristic period.*

8. Variational construction for the distribution functional

We have, in principle, determined moments which satisfy the necessary and sufficient statistical condition (4.6). The complete perturbation theory solution of the statistical problem requires the construction of a distribution functional which yields these second-, third-, and fourth-order moments and fixes the higher moments. A proof of the realizability of such a functional would, of course, be the most satisfying check on the validity of our method. It was indicated earlier that the distribution functional certainly is not fully determined by its second-, third-, and fourth-order moments. We need an additional condition to pick out a particular distribution functional appropriate to the physical situation. We shall adopt, for this purpose, a variational procedure which defines and determines the most nearly normal distribution consistent with the given moments, and with any additional physical conditions which can be expressed in terms of distribution moments.

The variational method is best formulated in terms of the transform functional Φ. We therefore shall summarize the foregoing perturbation theory

*In this connection we should note that in a physical situation the driving forces exerted on low wave number modes (whose behavior is substantially influenced by direct action of the driving forces) will exhibit characteristic periods comparable to those induced by the nonlinear forces exerted by the interaction with the higher wave numbers. This is true, for example, in turbulence supported by a steady shear flow.

in terms of frequency transforms. In order to deal with a denumerably infinite set of variables we choose an arbitrarily long time interval T and define the series transform $\tilde{z}(\omega)$ of any time function $z(t)$ by

$$(8.1) \qquad z(t) = T^{-1} \sum_{\omega} \tilde{z}(\omega) e^{-i\omega t}, \qquad \tilde{z}(\omega) = \int_T z(t) e^{+i\omega t} dt.$$

(The transform of any quantity will be represented by the corresponding quantity with a wavy overbar.) In the limit $T \to \infty$, the series transform defined in this way goes over, without further normalization, into the integral transform defined by $z(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{z}(\omega) e^{-i\omega t} d\omega$, provided $\tilde{z}(\omega)$ exists in the limit. The transform of the averaged response of mode q_α to an infinitesimal perturbation $h_\alpha(t)$ is

$$(8.2) \qquad \delta \tilde{q}_\alpha(\omega) = \tilde{g}_{\alpha\alpha}(\omega) \tilde{h}_\alpha(\omega),$$

where $\tilde{g}_{\alpha\alpha}(\omega)$ is the transform of $g_{\alpha\alpha}(\tau)$. The expressions for the perturbation-theory moment values become simpler, in some respects, in the transformed representation. For example, the transform of (5.8) is

$$(8.3) \qquad \tilde{\tilde{G}}_{\alpha\alpha}(\omega) = \tilde{\tilde{g}}_{\alpha\alpha}(-\omega) \tilde{\tilde{F}}_{\alpha\alpha}(\omega).$$

We shall choose as fundamental arguments for Φ the real and imaginary parts of the quantities $\tilde{q}_\alpha(\omega)$, $\tilde{f}_\alpha(\omega)$ for $\omega > 0$. (The corresponding variables for $\omega < 0$ then are fixed by reality requirements.) For simplicity in notation let us denote all these independent real and imaginary parts by the elements y_n of a row vector $\underset{\sim}{y}$, so that $\Phi = \Phi(\underset{\sim}{y})$. We may assume from the physics of our problem that frequencies above some arbitrarily high limit will be negligibly excited. We may therefore cut off the sequence y_n at some arbitrarily high n and deal with a finite number of variables.

Let us write

$$(8.4) \qquad \Phi(\underset{\sim}{y}) = \phi^*(\underset{\sim}{y})\phi(\underset{\sim}{y}),$$

which expresses the non-negativity of Φ, and otherwise involves no loss of generality. We define the most normal distribution consistent with given fixed moment values as that distribution for which the probability amplitude ϕ

minimizes

$$(8.5) \qquad I = \frac{\int \nabla_{\underset{\sim}{y}}\phi^* \cdot \nabla_{\underset{\sim}{y}}\phi \, d\underset{\sim}{y}}{\int \phi^* \phi \, d\underset{\sim}{y}}$$

under the integral constraints represented by the fixed moment values, and the boundary condition $\phi(\infty) = 0$.* In our problem, the fixed moments are all the second, third, and fourth moments of the y_n, as given by the perturbation theory. Therefore the ϕ so defined obeys the Schroedinger equation

$$(8.6) \qquad \nabla_y^2\phi + \left[\varepsilon - \sum_{n,m} \lambda_{nm} y_n y_m - \sum_{n,m,\ell} \lambda_{nm\ell} y_n y_m y_\ell - \sum_{n,m,\ell,s} \lambda_{nm\ell s} y_n y_m y_\ell y_s\right]\phi = 0,$$

where ε is the lowest eigenvalue for which the equation has a solution vanishing at infinity, and the Lagrange multipliers λ_{nm}, $\lambda_{nm\ell}$, $\lambda_{nm\ell s}$ are determined *a posteriori* by the specified moment values. It is apparent that we may take ϕ real without loss of generality.

It is readily verified that if the specified moments had consisted only of all the second moments ($\lambda_{nm\ell} = 0$, $\lambda_{nm\ell s} = 0$), the Φ defined by this criterion would have been the joint normal distribution yielding these moments. (The Schroedinger equation for this case describes an arbitrarily oriented multi-dimensional harmonic oscillator.)

The solution of (8.6) in a space of an indefinitely large number of dimensions would be impracticable except for the fact that the skew moments representing departure of the distribution from multivariate normal are small, so that the individual nondiagonal potential terms $\lambda_{nm\ell} y_n y_m y_\ell$, $\lambda_{nm\ell s} y_n y_m y_\ell y_s$ in (8.6) may be regarded as small perturbations (with suitable precautions and subject to complications). In this case, it may be seen to follow from the hypothesis of weak dependence that ϕ satisfying (8.6) gives a nearly independent-Gaussian distribution for any few modes when averaged over the distribution of the other modes. Then the perturbation solution of (8.6) may be written

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*An alternate variational criterion is the minimization of $\int d\underset{\sim}{y}\Phi \ln(\Phi/\int \Phi d\underset{\sim}{y})/\int \Phi d\underset{\sim}{y}$ under the same integral constraints. This provides a direct analogy to the principle of equal *a priori* probabilities based on the H-theorem, but it must be remembered that Φ is not the Gibbs probability density and the appropriateness of this principle is not self-evident in the present context. Although the alternate criterion leads more readily to explicit expressions for Φ, that presented in the text has a certain intrinsic interest because of the appearance of the probability amplitude in a classical problem. It is a consequence of the weak dependence property that both criteria lead to nearly the same distribution; the difference does not seem significant, particularly in view of the irreducible arbitrariness in assigning statistics to the external driving forces.

$$(8.7)\quad \phi(\underset{\sim}{y}) = \prod_{n,m,\ell} N_{nm\ell}^{1/2} \left(1 + \tfrac{1}{2} c_{nm\ell} y_n y_m y_\ell\right) \prod_{r,s,t,u} N_{rstu}^{1/2} \left(1 + \tfrac{1}{2} d_{rstu} y_r y_s y_t y_u\right) \chi(\underset{\sim}{y}),$$

where $\chi(\underset{\sim}{y})$ is a joint normal distribution amplitude and the N's are normalization factors. The $c_{nm\ell}$, d_{rstu}, and the parameters in $\chi(\underset{\sim}{y})$ must be chosen to give the perturbation theory moment values. A sufficient condition for the realizability of our moment values is the existence of such c's and d's.

We shall reserve for a subsequent paper the evaluation of the parameters, the discussion of consistency questions, and the method of use of the distribution functional. We wish to point out here that, assuming the success of the procedure, the accuracy of the predicted higher moments may be expected to become poor when the order of the moment is comparable to some function of the total number of degrees of freedom q_a which are excited. This is because our underlying assumption of weak statistical dependence cannot be asserted for such high-order moments.

We must also postpone discussion of the interesting question of the close relation of distributions of the form defined by (8.6) to the M-B distribution of simultaneous values, in the non-dissipative case, and to the condition of minimum entropy production in the dissipative case.

9. Isotropic turbulence

9.1 Zero magnetic field. We shall now present, in terms of the usual spectral representation, the closed equations for the velocity covariance which result for isotropic nonmagnetic turbulence, under the simplifying assumption (7.5). Without loss of generality, we may take the external force field divergenceless. Then we may define the scalar functions U, F, and G by

$$(9.1)\qquad \langle u_i(\underset{\sim}{k},t+\tau) u_j^*(\underset{\sim}{k},t)\rangle = \tfrac{1}{2} P_{ij}(\underset{\sim}{k})\, U(k,\tau),$$

$$\langle F_i(\underset{\sim}{k},t+\tau) F_j^*(\underset{\sim}{k},t)\rangle = \tfrac{1}{2} P_{ij}(\underset{\sim}{k})\, F(k,\tau),$$

$$\mathrm{Re}\{\langle F_i(\underset{\sim}{k},t+\tau) u_j^*(\underset{\sim}{k},t)\rangle\} = \tfrac{1}{2} P_{ij}(\underset{\sim}{k})\, G(k,\tau). \qquad (\mathrm{Re}\{\} \text{ denotes 'real part'})$$

These scalars all are real, and U and F are even functions of τ.* $U(k,\tau)$ and

*Rigorous isotropy requires $L \to \infty$, with our boundary conditions. We reserve for a more detailed paper the discussion of the validity of the passage to conditions (9.1), and the associated passage to the Fourier integral representation used in (9.8).

$F(k,\tau)$, suitably normalized, are the spatial Fourier coefficients of the defining scalars of the velocity and driving force spacetime autocovariances, and $k^2U(k,0)$ is proportional to the usual one-dimensional wave number spectrum function.

We may obtain from (3.1) (with $\underset{\sim}{w} = 0$) the equation

$$(9.2) \qquad \dot{U}(k,\tau) + \nu k^2 U(k,\tau) = S(k,\tau) + G(k,\tau),$$

where

$$(9.3) \qquad S(k,\tau) = \mathrm{Im}\left\{k_\ell \sum_{\underset{\sim}{k}'} < u_j(\underset{\sim}{k}',t)\, u_\ell\,(\underset{\sim}{k}-\underset{\sim}{k}',t)\, u_j^*(\underset{\sim}{k},t-\tau) > \right\},$$

and Im $\{\ \}$ denotes imaginary part. The quantity $k^2S(k,0)$ is proportional to the total net rate of energy transfer by nonlinear interaction to all the modes of wave number k. Evaluation of $S(k,\tau)$ and $G(k,\tau)$ in terms of $U(k,\tau)$ and $F(k,\tau)$ may be carried out directly on the basis of the preceding analysis, since the $u_i(\underset{\sim}{k},t)$ are linear homogeneous functions of the q_α. Adopting (7.5) we have the expression

$$(9.4) \qquad g(k,\tau) = \frac{U(k,\tau)}{U(k,0)} \exp(-\nu k^2 \tau), \qquad (\tau > 0),$$

for the diagonal impulse response of a mode of wave number k. According to the perturbation and weak dependence hypotheses, then,

$$(9.5) \qquad G(k,\tau) = \mathrm{Re}\,\{< F_i(\underset{\sim}{k},\, t+\tau)\overline{\delta u}_i^*(\underset{\sim}{k},t) >\}, \qquad (t \to \infty),$$

where

$$(9.6) \qquad \overline{\delta u}_i^*(\underset{\sim}{k},t) = \int_0^t g(k,t-s)\, F_i^*(\underset{\sim}{k},s)\, ds.$$

Proceeding as before, we find*

$$(9.7) \qquad G(k,\tau) = \int_0^\infty ds\, \frac{U(k,s)}{U(k,0)} \exp(-\nu k^2 s)\, F(k,s+\tau)$$

The evaluation of $S(k,\tau)$ is more involved because of complicated angle-averages which occur. The result, expressed in terms of a Fourier integral

*$\underset{\sim}{F}(\underset{\sim}{k},t)$ and $\underset{\sim}{u}(\underset{\sim}{k},t)$ are each linear combinations of several independent variables; (4.7) ensures that no consequent cross terms enter (9.7) and (9.8).

representation,* is

$$(9.8)\qquad S(k,\tau) = \pi k \int_0^\infty ds \int_0^\infty k'dk' \int_{|k-k'|}^{k+k'} k''\,dk'' \left[\frac{a(k,k',k'')}{U(k,0)} \exp(-\nu k^2 s) U(k,s) U(k',s+\tau) \right.$$

$$\left. \times U(k'',s+\tau) - \frac{b(k,k',k'')}{U(k',0)} \exp(-\nu k'^2 s) U(k,s-\tau) U(k',s) U(k'',s) \right],$$

where

$$(9.9)\qquad a(k,k',k'') = (1 - xyz - 2y^2z^2),$$

$$b(k,k',k'') = 2k^{-1}k'(-z + xy + x^2z + y^2z + 2z^3 + 2xyz^2),$$

and x,y,z are the cosines of the interior angles opposite k, k', k", respectively, in a triangle with these quantities as sides. The quantities a, b obey the relations

$$(9.10)\qquad a(k,k',k'') = a(k,k'',k') \geq 0,$$

$$k^2 b(k,k',k'') = k'^2 b(k',k,k''),$$

$$b(k,k',k'') + b(k,k'',k') = 2a(k,k',k''),$$

in the domain for which they are defined. The integration indicated in (9.8) actually is symmetric between k' and k", since, if $|k - k'| \leq k'' \leq k + k'$, then $|k - k''| \leq k' < k + k''$. As a consequence, it may be seen that (9.10) expresses the consistency of the expression for S(k,0) given by (9.8) with the detailed conservation property (3.10).** For many purposes, a and b may be approximated by very simple forms.

Equations (9.2),(9.7),(9.8), together with the boundary condition $U(k,\infty)=0$, form a closed system determining $U(k,\tau)$ in terms of $F(k,\tau)$, the driving force

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*See footnote following equation (9.1). The normalization of the integral transforms is fixed by $U(k,\tau) = \int \hat{U}(x,\tau) e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} d^3x$, where $\hat{U}(x,\tau) = \langle \hat{\underset{\sim}{u}}(\underset{\sim}{y}+\underset{\sim}{x},\ t+\tau)\cdot\hat{\underset{\sim}{u}}(\underset{\sim}{y},t) \rangle$, with similar definitions for $G(k,\tau)$, $F(k,\tau)$ when interpreted as integral transforms. This implies a corresponding renormalization of $\underset{\sim}{u}(\underset{\sim}{k},t)$, $\underset{\sim}{F}(\underset{\sim}{k},t)$ to unit volume.

**The integrand term in (9.8) involving a( ) indicates a positive contribution to S(k,0) from all triangle interactions among modes of wave numbers k,k',k", while that involving b( ) indicates a negative contribution. Equation (9.10) assures that the contributions to $k^2S(k,0)$, $k'^2S(k',0)$, $k''^2S(k'',0)$ from these same interactions add to zero as required by detailed conservation. The factors $1/U(k,0)$ in the positive term of S(k,0) and $1/U(k',0)$ in the negative term indicate that the theory gives a net energy-transfer in the direction tending to establish equipartition among the Fourier modes.

covariance. The detailed discussion of these equations will be reserved for a subsequent paper, but we shall indicate very briefly here the consistency with them of Kolmogorov's similarity considerations.[2],[4] The Kolmogorov theory of very high Reynolds numbers predicts the existence of an inertial range of wave numbers in which dissipation is negligible and for which

$$U(k,0) = (k/k_1)^{-11/3}\, U(k_1,0), \tag{9.11}$$

if both k and k_1 are in this range. This spectrum form follows firmly from dimensional analysis if it can be established that energy transfer in the inertial range proceeds by a cascade process through a chain of successively higher wave numbers, the essential requirement being that no appreciable fraction of the energy is transferred directly from energy-containing to dissipation range.

If the nonlinear interaction does produce such a cascade process, it is clear that the characteristic time associated with a mode in the inertial range will have a value proportional to $k^{-2/3}$. This result may be obtained by forming a time from the length k^{-1} and the rms velocity corresponding to the portion of the total energy which is contained in a band of width k centered on k. According to the Kolmogorov dimensional considerations, it follows that in the inertial range $U(k,\tau)$ must be of the form

$$U(k,\tau) = c\varepsilon^{2/3}\, k^{-11/3}\, r(\varepsilon^{1/3}k^{2/3}\tau), \tag{9.12}$$

where c is a numerical constant, ε is the total rate of energy transfer/unit mass by the assumed cascade process, and $r(\ )$ is a universal function normalized by $r(0) = 1$.

We may now verify from (9.8) that (9.12) is consistent with the existence of an energy cascade process. We consider the contribution to $S(k_1,0)$ due to interactions with all the modes of wave number $k \geq k_2 \gg k_1$, assuming both k_1 and k_2 are well within the inertial range. This is essentially equivalent to considering the contribution to $S(k_1,0)$ due to integration over the region $k' \geq k_2$ (for which the k" integration is confined to $k" \sim k'$). Noting from (9.12) that $U(k',0) \ll U(k_1,0)$ if $k' \geq k_2$, then using (9.10) to estimate the relative magnitudes of the two terms of (9.8), and neglecting the exponential factors in accordance with the assumed smallness of the dissipation, we may verify that this contribution is

$$-\left[S(k_1,0)\right]_{>k_2} \sim \frac{3\pi}{2}\, c^2 \varepsilon k_1^{-3} \left[\int_0^\infty (r(x))^2 dx\right] \left(\frac{k_2}{k_1}\right)^{-4/3}. \tag{9.13}$$

Integrating (9.13) over all modes from the bottom of the inertial range (wave number k_{min}), which we assume is well below k_1, we have

$$(9.14) \qquad -4\pi \int_{k_{min}}^{k_1} [S(k,0)]_{>k_2} \, k^2 dk \sim c'\varepsilon \left(\frac{k_2}{k_1}\right)^{-4/3},$$

where c' is a new numerical constant.

This result indicates that if k_2/k_1 is sufficiently large there is negligible direct transfer of energy from modes of wavenumber $< k_1$ to modes of wavenumber $> k_2$. Thus the asymptotic validity of the cascade concept is corroborated.

Further analysis shows that c' in (9.14) is of order unity. This indicates that the number of distinct cascade steps in the entire inertial range is of the order of magnitude of $\log_{10}(k_{max}/k_{min})$, where k_{max} defines the top of the range. The similarity theory for high Reynolds number indicates the rough relation $\ln(k_{max}/k_{min}) \sim \frac{3}{4} \ln(Re)$, where Re is the Reynolds number based on macroscale and rotal rms velocity. We may estimate crudely that for $Re \sim 10^6$ there would be only about four distinct steps.

The rather ill-defined nature of the cascade process suggests that even in quite low Reynolds number turbulence the modes in the energy-containing range interact with many other modes. This adds support to the validity of the weak dependence hypothesis in this case.

9.2 Non-zero magnetic field. Extending our results to isotropic hydromagnetic turbulence, we find

$$(9.15) \qquad \dot{W}(k,\tau) + \bar{\nu}k^2 W(k,\tau) = Q(k,\tau),$$

$$(9.16) \qquad \dot{U}(k,\tau) + \nu k^2 U(k,\tau) = S(k,\tau) + \tilde{Q}(k,\tau) + G(k,\tau),$$

where

$$(9.17) \qquad \langle w_i(\underline{k},t+\tau) w_j^*(\underline{k},t) \rangle = \frac{1}{2} P_{ij}(\underline{k}) \, W(k,\tau),$$

$$(9.18) \qquad Q(k,\tau) = \pi k \int_0^\infty ds \int_0^\infty k'dk' \int_{|k-k'|}^{k+k'} k''dk'' \Big[\frac{d(k,k',k'')}{W(k,0)} \exp(-\bar{\nu}k^2 s) W(k,s)$$

$$\times W(k',s+\tau) U(k'',s+\tau) - \frac{h(k,k',k'')}{W(k',0)} \exp(-\bar{\nu}k'^2 s) W(k,s-\tau) W(k',s)$$

$$\times U(k'',s) - \frac{j(k,k',k'')}{U(k',0)} \exp(-\nu k'^2 s) W(k,s-\tau) W(k'',s) U(k',s) \Big],$$

$$(9.19)\qquad \tilde{Q}(k,\tau) = \pi k \int_0^\infty ds \int_0^\infty k'dk' \int_{|k-k'|}^{k+k'} k''\,dk'' \left[\frac{a(k,k',k'')}{U(k,0)} \exp(-\nu k^2 s) U(k,s) W(k',s+\tau) \right.$$

$$\left. \times W(k'',s+\tau) - \frac{c(k,k',k'')}{W(k',0)} \exp(-\nu k'^2) U(k,s-\tau) W(k',s) W(k'',s) \right].$$

The new angular dependence factors are given by

$$(9.20)\qquad c(k,k',k'') = 2k^{-1}k'\, z(1-y^2)$$

$$d(k,k',k'') = 2(1 + xyz)$$

$$h(k,k',k'') = 2k^{-1}k'(1 + xy + y^2 z - z^2)$$

$$j(k,k',k'') = 2k^{-1}k'(z - xy - x^2 z - 2xyz^2)$$

$U(k,\tau)$ has its previous meaning, and both $S(k,\tau)$ and $G(k,\tau)$ are given by the previous expressions. We note that $k^2 Q(k,0)$ is proportional to the net rate of energy transfer to all magnetic modes of wave number k as a result of the nonlinear interaction.

10. Discussion

The fundamental condition for the validity of the theory we have presented is that each excited mode interact with sufficiently many others that the effect of each elementary interaction can be considered a small perturbation. We have pointed out that this condition should be met even for the energy-containing range of quite moderate Reynolds number turbulence. If our subsequent perturbation hypothesis is justified, as indicated by the evidence presented in Section 5, then the principal limitation to the physical applicability of the theory would appear to lie in the artificiality of our treatment of the forces which supply energy to the turbulence.*

It would be more realistic to replace the f_α in (3.7) by terms giving the interaction of the fluctuating flow with, say, a steady shear flow. It may be seen that this would have the effect of introducing a linear coupling among the q_α, which would be incompatible with the homogeneity conditions (4.7). The precise nature of the driving forces should have comparatively little effect on the inertial and dissipation ranges (except for the residual anisotropy at high wave numbers), but it seems clear that a proper description of the energy-containing range in shear turbulence by our methods requires relaxation of the homogeneity condition

*The assumed homogeneous forces may be visualized as arising from a random volume distribution of stirring devices.

along at least one axis. The complications associated with this generalization are more severe, in certain respects, than those associated with the relaxation of stationarity required for the treatment of the decay of homogeneous turbulence. Other generalizations of interest are the inclusion of external magnetic fields and compressibility effects.

It may be of value to re-emphasize the difference between the hypothesis of weak dependence, introduced in Section 4, and the hypothesis employed by Heisenberg [10] and in somewhat different form by Chandrasekhar [11], that the second and fourth moments of the (two-time) distribution of the spatial Fourier components of the fields are related as in an independent distribution. For brevity, we shall confine discussion to the idealized dissipationless system discussed in Section 6. Setting ν_α, $f_\alpha = 0$, we may obtain readily from (3.7) the relation

$$(10.1) \qquad \dot{R}_{\alpha\alpha}(0) = \sum_{\beta,\gamma,\mu,\lambda} A_{\alpha\beta\gamma} A_{\alpha\mu\lambda} \int_0^\infty T_{\mu\lambda\beta\gamma}(s)ds.$$

If, now, we assert Heisenberg's hypothesis, we obtain

$$(10.2) \qquad \dot{R}_{\alpha\alpha}(0) = 2 \sum_{\beta,\gamma} (A_{\alpha\beta\gamma})^2 \int_0^\infty R_{\beta\beta}(s)R_{\gamma\gamma}(s)ds,$$

where (3.11) and (4.7) have been used. It follows from the properties of correlation functions that the integrals on the right side of (10.2) are positive quantities, so that this hypothesis leads here to the impossible result $\dot{R}_{\alpha\alpha}(0) > 0$.

The essential error in (10.2) is the neglect of the skew elements $T_{\mu\lambda\beta\gamma}(s)$ (μ,λ,β,γ not equal in pairs) which occur in (10.1). Although these terms individually are very small, their number is greater than that of the non-skew elements by a factor the order of the total number of degrees of freedom of the system. The neglect of their cumulative contribution is <u>not</u> justified on the basis of our weak dependence hypothesis. Quite a different matter is the assertion $T_{\beta\gamma\beta\gamma}(s) \sim R_{\beta\beta}(s)R_{\gamma\gamma}(s)$ for the non-skew element appearing in (5.3); this approximation follows legitimately from the weak dependence hypothesis. In distinction from the incorrect result above, our (6.12) yields $\dot{R}_{\alpha\alpha}(0) = 0$.

The inconsistencies associated with the assumption of normal relation between second and fourth moments are discussed more fully in [12]; it is shown there that the implied violation of the equations of motion is a serious one.

The immediate program following from the work presented here involves the determination of the spectrum in wave number and frequency of isotropic hydromagnetic turbulence from equations of the type presented in Section 9, and examination of the energy transfer among the various kinetic and magnetic modes. This program entails validation of assumption (7.5), and the development of systematic approximation schemes for the solutions of the difficult nonlinear integro-differential equations of motion for the covariances.

The more fundamental program suggested by the work reported lies in the investigation of the distribution functional defined in Sections 4 and 8, further development of the statistical mechanical method there outlined, and the application of this method to other problems of the many-body type. Of particular interest is the applicability of the present approach to quantum statistical mechanics. The time-dependent Schroedinger equation of a quantum mechanical system constitutes a set of first-order equations of motion for the orthogonal components of the state-vector in any given representation. In problems where an appropriate perturbation treatment is valid one may obtain equations, analogous to those derived in Sections 6 and 7, which determine the correlation and response matrices of the orthogonal components.

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Addendum to Section 9

We shall present here the full set of equations determining the impulse response functions and covariance scalars when assumption (7.5) is not made. Consider first non-magnetic turbulence. In place of (9.7) we have, in the general case,

$$G(k,\tau) = \int_0^\infty g(k,s)\, F(k,s+\tau)\,ds \,, \tag{A.1}$$

and in place of (9.8) we have

$$S(k,\tau) = \frac{k^2}{4} \int_0^\infty ds \int d^3k' \Big[a(k,k',|\underset{\sim}{k}-\underset{\sim}{k}'|)\, g(k,s)U(k',s+\tau)U(|\underset{\sim}{k}-\underset{\sim}{k}'|,s+\tau) - b(k,k',|\underset{\sim}{k}-\underset{\sim}{k}'|)U(k,s-\tau)g(k',s)U(|\underset{\sim}{k}-\underset{\sim}{k}'|,s)\Big] \,, \tag{A.2}$$

where we also have re-expressed the integration over wave vector space. To obtain a complete set of equations we must now express (7.3) for the isotropic case. From (3.1) we may obtain the equation

$$\dot{G}(k,\tau) + \nu k^2 G(k,\tau) = H(k,\tau) + F(k,\tau) \,, \tag{A.3}$$

where

$$H(k,\tau) = \mathrm{Im}\left\{k_l \sum_{\underset{\sim}{k}'} \left\langle u_j(\underset{\sim}{k}',t)\, u_l(\underset{\sim}{k}-\underset{\sim}{k}',t) F_j^*(\underset{\sim}{k},t+\tau)\right\rangle\right\} \,. \tag{A.4}$$

Evaluating $H(k,\tau)$ by our perturbation procedure, and following the argument by which (7.3) was obtained from (7.1), we find

$$\dot{g}(k,\tau) + \nu k^2 g(k,\tau) = -\frac{k^2}{4} \int_0^\tau ds \int d^3k'\, b(k,k',|\underset{\sim}{k}-\underset{\sim}{k}'|)\; g(k,\tau-s)\, g(k',s)\, U(|\underset{\sim}{k}-\underset{\sim}{k}'|,s), \quad (\tau > 0) \,, \tag{A.5}$$

upon making the transformation to the Fourier integral representation. Equations (9.2), (A.5), (A.1), and (A.2) now form a complete set which determine $g(k,\tau)$ and $U(k,\tau)$ under the boundary conditions $g(k,+0) = 1$, $U(k,\infty) = 0$.

We shall denote the impulse response function for magnetic modes by $g_m(k,\tau)$. Instead of (9.18), (9.19), we have in the general case

$$(A.6)\quad Q(k,\tau) = \frac{k^2}{4}\int_0^\infty ds \int d^3k' \Big[d(k,k',|\underline{k}-\underline{k}'|)g_m(k,s)W(k',s+\tau)U(|\underline{k}-\underline{k}'|,s+\tau)$$
$$- h(k,k',|\underline{k}-\underline{k}'|)W(k,s-\tau)g_m(k',s)U(|\underline{k}-\underline{k}'|,s)$$
$$- j(k,k',|\underline{k}-\underline{k}'|)W(k,s-\tau)g(k',s)W(|\underline{k}-\underline{k}'|,s)\Big],$$

and

$$(A.7)\quad \tilde{Q}(k,\tau) = \frac{k^2}{4}\int_0^\infty ds \int d^3k' \Big[a(k,k',|\underline{k}-\underline{k}'|)g(k,s)W(k',s+\tau)W(|\underline{k}-\underline{k}'|,s+\tau)$$
$$- c(k,k',|\underline{k}-\underline{k}'|)U(k,s-\tau)g_m(k',s)W(|\underline{k}-\underline{k}'|,s)\Big],$$

while the equations for $g(k,\tau)$, $g_m(k,\tau)$ are

$$(A.8)\quad \dot{g}(k,\tau) + \nu k^2 g(k,\tau) = n(k,\tau) + \tilde{m}(k,\tau), \qquad (\tau > 0),$$

$$(A.9)\quad \dot{g}_m(k,\tau) + \bar{\nu} k^2 g_m(k,\tau) = m(k,\tau), \qquad (\tau > 0),$$

where

$$(A.10)\quad \tilde{m}(k,\tau) = -\frac{k^2}{4}\int_0^\tau ds \int d^3k' c(k,k',|\underline{k}-\underline{k}'|)g(k,\tau-s)g_m(k',s)W(|\underline{k}-\underline{k}'|,s),$$

$$(A.11)\quad m(k,\tau) = -\frac{k^2}{4}\int_0^\tau ds \int d^3k' g_m(k,\tau-s)\Big[h(k,k',|\underline{k}-\underline{k}'|g_m(k',s)U(|\underline{k}-\underline{k}'|s)$$
$$- j(k,k',|\underline{k}-\underline{k}'|)g(k',s)W(|\underline{k}-\underline{k}'|,s)\Big],$$

and $n(k,\tau)$ is the right side of (A.5).

The equations presented for the impulse response functions have a simpler structure than the equations for the covariance scalars. Thus, if $U(k,\tau)$ were a known function, $g(k,\tau)$ could be computed from (A.5) by straightofrward step-by-step integration from the point $\tau = 0$.

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